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On singleton composites in non-compact WZW models

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Abstract: We examine the $\widehat{\mathfrak{so}}(2, D-1)$ WZW model at the subcritical level $-(D-3)/2$. It has a singular vacuum vector at Virasoro level 2. Its decoupling constitutes an affine extension of the equation of motion of the $(D+1)$-dimensional conformal particle, i.e. the scalar singleton. The admissible (spectrally flowed) representations contain the singleton and its direct products, consisting of massless and massive particles in AdS_D. In $D = 4$ there exists an extended model containing both scalar and spinor singletons of $\mathfrak{sp}(4)$. Its realization in terms of 4 symplectic-real bosons contains the spinor-oscillator constructions of the $4D$ singletons and their composites. We also comment on the prospects of relating gauged versions of the models to the phase-space quantization of partonic branes and higher-spin gauge theory.

Keywords: Space-Time Symmetries, Conformal Field Models in String Theory, Field Theories in Lower Dimensions, AdS-CFT Correspondence.

JHEP02(2007)097

Contents

1. Introduction

At the core of Quantum Field Theory (QFT) lies the representation theory of non-compact Lie algebras. Their non-trivial unitary irreducible representations (UIR) are necessarily infinite-dimensional, with energy bounded from either below or above, corresponding, in the first-quantized language, to the presence of particles and anti-particles. In this paper, we shall examine this feature in the $\widehat{\mathfrak{so}}(2, D-1)$ Wess-Zumino-Witten (WZW) model with subcritical level $k = -(D-3)/2$, with particular focus on the case of $D = 4$.

Non-compact WZW models furnish a novel mathematical topic that remains relatively uncharted in comparison with the compact case, essentially due to the above-mentioned

JHEP02(2007)097

fact that infinite-dimensional UIRs arise already at the level of the finite-dimensional subalgebra. Previous investigations related to QFT in anti-de Sitter (AdS) spacetime have considered cases with *non-critical* level $k + h^\vee < 0$ [1–4], *critical* level $k + h^\vee = 0$ [5, 6] as well as *subcritical* levels $k + h^\vee > 0$ [7], where $h^\vee$ is the dual Coxeter number given by $h^\vee = D - 1$ for $\mathfrak{so}(2, D-1)$.

The attempts of interpreting the critical case as a form of tensionless limit have been problematic, technically, at the level of two-dimensional conformal field theory (CFT), where the standard Sugawara construction breaks down. In this respect, the subcritical cases are more tractable. Interestingly, already in [7] it was found that in $D = 4$ the subcritical $k' = -5/2$ Verma module built on the scalar singleton exhibits a large number of singular vectors, signalling some form of symmetry enhancement (one might also speculate about some form of duality between subcritical models with levels k and k' obeying $k+k' = D - 1$). More recently, starting with the work in [8], attempts to formulate string theory duals of $\mathcal{N} = 4$ Super-Yang-Mills theory have led to CFTs [9–11] that are tantalizingly close to our subcritical cases, although the precise relation remains to be seen. $\widehat{\mathfrak{so}}(2, D-1)$ WZW models in four dimensions have also been considered in the literature and, for the choice $D = 5$, it has been proven that a dynamical sector that is Einstein's general relativity arises [12].

Our physical interest in the subcritical models with $k = -(D-3)/2$ (in the case of purely bosonic models) stems from the fact that they exhibit *compositeness*, whereby physical particles, such as photons and gravitons, instead of being fundamental, are made up from more elementary constituents, known as *singletons*. Drawing on the group theoretical underpinning of the AdS/CFT correspondence and the features of Vasiliev's unfolded approach to Higher-Spin Gauge Theory (HSGT) and QFT in general (for a recent review, see [13] and references therein; for a recent development in the direction of M-theory see [14]), one arrives at the idea that the singletons are the basic constituents of String Field Theory (SFT) and that this feature is manifest when SFT is expanded around tensionless AdS backgrounds — while being blurred in expansions around tensionful backgrounds including flat spacetime.

A key step in entertaining this partonic picture is to find a connection between the standard background-dependent quantization of strings and a covariant formulation that combines phase-space quantization of discretized tensionless strings with unfolded dynamics. Roughly speaking, the worldsheet correlators — which are normally expanded around metric backgrounds with finite string tension in terms of cut-off field-theoretic Feynman diagrams – should be given an expansion around a topological vacuum with zero tension, resulting in algebraic structures making up the internal sector of unfolded SFT. In the unfolded approach, the total manifold is a non-commutative fiber (the string phase space) times a commutative base manifold (containing the actual physical spacetime), and the unfolded field equation is the (non-linear) cohomology of the phase-space BRST operator plus the exterior derivative. As it is the case already in HSGT, the projection to the base then yields a Free Differential Algebra (FDA) constituting a quasi-topological QFT with infinitely many zero-forms. Here, spacetime, instead of being a slice of the non-commutative phase space, reemerges, fully covariantly, upon fixing the manifest homotopy invariance of

JHEP02(2007)097

the FDA.

Put somewhat differently, we have in mind a worldsheet duality between, on the one hand, the standard BRST formulation based on the Virasoro algebra and sewing of surfaces, and, on the other hand, a non-standard BRST formulation (yet to be spelled out in detail) based on gauging subcritical affine algebras and sewing of string partons. As we shall see in this paper, and which is of great conceptual interest, the subcritical affine algebras incorporate the key features of the corresponding, already known, higher-spin enveloping algebras, thus, seemingly, superseding the latter as the key algebraic structure underlying unfolded SFT.

In a previous paper, [15], two of the authors of the present paper have examined ordinary bosonic branes in AdS_D using four related models: *i)* the normal-coordinate expansion around Nambu-Goto solitons describing rotating branes; *ii)* conformal branes close to the boundary of the AdS spacetime; *iii)* discretized tensionless branes on Dirac's hypercone; and *iv)* a topological non-compact gauged subcritical WZW model (closed singleton strings) in $D = 7$. It was found that *(i)-(iii)* realize the *brane partons as scalar singletons*. The same was conjectured to hold also in the WZW model *(iv)*, and, as already announced, this seems indeed to be the case, as we shall demonstrate in this paper (see section 3). Out of these models, we expect the WZW-model formulation to be the most viable one, since it entails the incorporation of a "stringy" multi-parton spectrum — with massless as well as massive representations — into the above-mentioned unfolded approach to QFT. That is, as outlined above, the WZW model realizes an algebra of functions on a phase space (given, roughly, by the direct sum over direct products of singleton phase spaces) to be identified as the fiber of an unfolded formulation of SFT, in which all gauge symmetries (on the base manifold), including higher-spin symmetries, are unbroken.

The affine realization of the singletons and its composites in the case of $\widehat{\mathfrak{so}}(2, D-1)$ rests on the simple observation that, for the *subcritical level*[1]

$$k = -\epsilon_0 \quad \equiv \quad -\frac{D-3}{2}\,, \tag{1.1}$$

there exists a singular vector at Virasoro level 2 in the NS sector whose decoupling plays the role of a non-perturbative equation of motion [16] selecting the physical operators. The singular vector amounts to the affine *hyperlight-likeness condition*

$$(M_A{}^C(z) M_{BC}(z)) - \text{trace} \sim 0\ , \tag{1.2}$$

where by $M_{AB}(z)$ we denote the currents. In section 3 we will discuss this condition in more detail and we will show that the physical operators of the subcritical WZW model are *twisted primary fields*: the twist, labeled by an integer $P \in \mathbb{Z}$, refers to that the lowest-weight conditions are shifted upward and downward P Virasoro levels for the KM charges with positive and negative AdS energy, respectively. As we shall see, the scalar singleton

[1] There is no coincidence that $-k$ is the energy of the singleton ground state. Interestingly, the subcritical case considered in [7], namely level $k' = -5/2$ in $D = 4$, is dual to our case in the sense that $k + k' = -h^\vee$. The precise relation remains to spelled out.

JHEP02(2007)097

($P = 1$) and anti-singleton ($P = -1$) and their $|P|$-fold tensor products arise naturally in this construction.

The singular vacuum vector induces a large number of singular vectors in the various P-twisted Verma modules. The model nonetheless contains non-unitary and non-topological states. Drawing on the continuum limit of discretized branes [15] as well as on the sigma-model description of null surfaces in anti-de Sitter spacetime [17] (see also [18, 19]), it makes sense to entertain the idea that the coset model based on

$$\frac{\widehat{\mathfrak{so}}(2, D-1)_{-\epsilon_0}}{\widehat{\mathfrak{so}}(D-1)_{-\epsilon_0} \oplus \widehat{\mathfrak{so}}(2)_{-\epsilon_0}}\,, \tag{1.3}$$

is a topological model consisting of symmetrized multipletons. In [15], a specific continuum limit of a discretized tensionless brane in AdS_D was shown, using a phase-space formulation on the $(D+1)$-dimensional ambient space, to be an Sp(2)-gauged free-field model with critical dimension $D = 7$. This model was then argued to be dual to the $\widehat{\mathfrak{so}}(6,2)_{-2}$ WZW model, and the gauging was imposed by hand in order to remove unphysical states. We shall describe the above gauging and its potential short-comings in more detail in [20].

In this paper we shall instead focus on the ungauged $\widehat{\mathfrak{so}}(2, D-1)_{-\epsilon_0}$ model. Eventually, we shall specify to the case of $\widehat{\mathfrak{so}}(2,3)_{-1/2}$ that can be realized in terms of 4 real symplectic bosons forming a quartet of $\mathfrak{sp}(4)$. The $2+2$ split of the bosons allows us to lift several results from the study of the $\widehat{\mathfrak{sp}}(2)_{-1/2}$ model in [21], with the important exception of the hyperlight-likeness condition (1.2), which couples the two doublets in a non-trivial fashion (the equation of motion of the $\widehat{\mathfrak{sp}}(2)_{-1/2}$ model is instead a 5-plet at level 4).

The paper is organized as follows: in section 2 we discuss the hyperlight-likeness condition and the special role of the scalar singletons in D dimensions. In section 3 we lift this condition to the non-compact WZW model and show how it is solved by the twisted primary fields. In section 4 we present the realization of the 4D model in terms of symplectic bosons and multiple sets of Heisenberg oscillators. In section 5 we compute the fusion rules using bosonization techniques. Finally, in section 6 we summarize our results.

2. Singletons and multipletons

Singletons are ultra-short unitary representations of $\mathfrak{so}(2, D-1)$ that admit no flat spacetime limit. They were first discovered in $D = 4$ by Dirac in 1963 [22]. His singletons may be realized [23, 24] as conformal particles on the 4-dimensional (singular) hypercone in 5-dimensional embedding space $\mathbb{R}^{2,3}$ with signature $(--+++)$. Gauging the worldline conformal group Sp(2) leaves a 4-dimensional physical phase space parameterized by two Heisenberg oscillators. The 10 oscillator bilinears generate a unitary representation of $\mathrm{Sp}(4) \simeq \mathrm{Spin}(2,3)$ in the Fock space, which decomposes into a scalar and a spinor representation consisting of the even and odd states, respectively. Dirac seemed intrigued by the fact that his representations do not survive the flat limit. This feature was later stressed by Flato and Frønsdal, who argued that, instead of limiting the formulation of Quantum Field Theory to flat spacetime, it might make more sense to start with a non-zero cosmological constant and recover the flat case in a limit. Dirac's representations could then play a role

as hidden, or internal, quantum variables. Indeed, a key property of AdS Quantum Field Theory, discovered by Flato and Frønsdal in [25], is that massless and massive one-particle states are *composites* consisting of two or more singletons (they introduced the term singleton referring to the fact that it occupies a single line in the non-compact weight space). In other words, the massless fermions and scalars, and even more interestingly, the photons and gravitons, which one would consider as fundamental in flat spacetime, are some form of two-singleton composites in AdS spacetime. One recognizes a similar reasoning behind Thorn's string bits and the more recent AdS/CFT correspondence. Flato and Frønsdal viewed, however, the singletons as "gauge" singlets as opposed to the "colored" string bits and fundamental fields of the holographic CFT's. One natural scenario in which singlet singletons may arise is in the partonic description of extended objects in AdS_D advocated in [15]. The main feature of this proposal is to combine the discretization with a $(D+1)$-dimensional phase-space formulation, without gauge fixing on the worldvolume, and then argue for enhanced sigma-model gauge symmetries in a combined tensionless and hypercone limit. The resulting gauge group contains one Sp(2) subgroup for each parton, that hence can be identified as a singleton realized as a $(D+1)$-dimensional conformal particle.

2.1 Elements of $\mathfrak{so}(2, D-1)$ representation theory

To give a group-theoretical presentation of singletons and their tensor products (see for example [26 – 30]) one starts from the hermitian $\mathfrak{so}(2, D-1)$ generators M_{AB} $(A, B = 0', 0, 1, \ldots, D-1)$ obeying[2]

$$[M_{AB}, M_{CD}] = 4i\eta_{BC}M_{AD}\,, \qquad M_{AB} \;=\; -M_{BA} \;=\; (M_{AB})^\dagger\,, \tag{2.1}$$

where $\eta_{AB} = \mathrm{diag}(-,-,+,\ldots,+)$. In terms of the AdS energy E and translation-boost generators $L_r^\pm$ $(r = 1, \ldots, D-1)$, defined by

$$E \;=\; M_{00'} \;=\; E^\dagger\,, \qquad L_r^\pm \;=\; M_{r0'} \mp iM_{r0} \;=\; (L_r^\mp)^\dagger\,, \tag{2.2}$$

and the spatial angular momenta $M_{rs} = -M_{sr} = (M_{rs})^\dagger$, the commutation rules assume the following form

$$[L_r^-, L_s^+] = 2\delta_{rs}E + 2iM_{rs}\,, \quad [E, L_r^\pm] \;=\; \pm L_r^\pm\,, \quad [M_{rs}, L_t^\pm] \;=\; 2i\delta_{st}L_r^\pm\,. \tag{2.3}$$

For $D \geq 4$ the physical representations are of lowest-weight type.[3] To describe these representations one starts from the Harish-Chandra module[4] $\mathfrak{V}(e_0, \mathbf{m}_0)$ (see e.g. [28]) generated by the repeated action of the energy-raising operators L_r^+ on a lowest-weight state $|e_0, \mathbf{m}_0; \ell\rangle$ obeying

$$L_r^-|e_0, \mathbf{m}_0; \ell\rangle \;=\; 0\,, \qquad (E - e_0)|e_0, \mathbf{m}_0; \ell\rangle \;=\; 0\,, \tag{2.4}$$

[2]We suppress Young projections of indices, which are always of unit strength, unless two sides of an equation have different index symmetries.

[3]The cases $D = 2$ and $D = 3$ require a separate analysis.

[4]Harish-Chandra modules are generalized Verma modules where by definition all null states generated by energy-lowering operators and spatial angular momenta are factored out.

and with ℓ denoting the weights of the UIR of $\mathfrak{so}(D-1)$ labeled by the highest weight $\mathbf{m}_0 = (m_0^1, m_0^2, \ldots, m_0^\nu)$, $m_0^1 \geq m_0^2 \geq \cdots \geq |m_0^\nu|$, where $\nu = [(D-1)/2]$ and $m_0^\nu \geq 0$ if D is even integer. The spectrum of the energy operator E acting in $\mathfrak{V}(e_0, \mathbf{m}_0)$ is bounded from below by the energy eigenvalue e_0 of the lowest-weight state, which one sometimes refers to as the ground state. The anti-linear inner product on the space of ground states yields an anti-linear inner product on $\mathfrak{V}(e_0, \mathbf{m}_0)$ in its turn inducing a maximal invariant subspace $\mathfrak{N}(e_0, \mathbf{m}_0) \subset \mathfrak{V}(e_0, \mathbf{m}_0)$. This maximal ideal consists of null states, i.e. states that are orthogonal to all states in $\mathfrak{V}(e_0, \mathbf{m}_0)$. It is nontrivial iff there exists at least one singular vector, namely a state $|e_0', \mathbf{m}_0'\rangle \in \mathfrak{V}(e_0, \mathbf{m}_0)$ with $e_0' > e_0$ obeying $L_r^-|e_0', \mathbf{m}_0'\rangle = 0$. The singular vector generates its own submodule of $\mathfrak{N}(e_0, \mathbf{m}_0)$. There may be several null submodules, possibly with additional substructure. Factoring out $\mathfrak{N}(e_0, \mathbf{m}_0)$ yields the non-degenerate lowest-weight space

$$\mathfrak{D}(e_0, \mathbf{m}_0) = \mathfrak{V}(e_0, \mathbf{m}_0)/\mathfrak{N}(e_0, \mathbf{m}_0) \ . \tag{2.5}$$

Every lowest-weight space can be flipped "upside down" into a highest-weight space,

$$\mathfrak{D}^-(e_0, \mathbf{m}_0) = \mathfrak{V}^-(e_0, \mathbf{m}_0)/\mathfrak{N}^-(e_0, \mathbf{m}_0) \, , \tag{2.6}$$

built on the highest-weight state

$$L_r^+|e_0, \mathbf{m}_0; \ell\rangle^- = 0 \, , \qquad (E + e_0)|e_0, \mathbf{m}_0; \ell\rangle^- = 0 \ . \tag{2.7}$$

We shall use the convention that superscript $+$ and $-$ indicate lowest and highest-weight spaces and states, with $+$ set as default value, and refer to the negative-energy states as anti-states. Defining the linear map

$$\pi\left(|e_0, \mathbf{m}_0\rangle^\pm\right) = |e_0, \mathbf{m}_0\rangle^\mp \, , \qquad \pi\left({}^\pm\langle e_0, \mathbf{m}_0|\right) = {}^\mp\langle e_0, \mathbf{m}_0| \ , \tag{2.8}$$

one can show that π lifts to a linear involutive $\mathfrak{so}(2, D-1)$ automorphism given by

$$\pi(M_{ab}) = M_{ab} \, , \qquad \pi(P_a) = -P_a \ . \tag{2.9}$$

where $M_{AB} = (M_{ab}, P_a)$, with $P_a = M_{a0'}$, $a = (0, r)$ being the space-time translations and M_{ab} the Lorentz transformations. It is also useful to introduce an anti-automorphism τ acting on states and generators as follows

$$\tau\left(|e_0, \mathbf{m}_0\rangle^\pm\right) = {}^\mp\langle e_0, \mathbf{m}_0| \, , \qquad \tau(M_{AB}) = -M_{AB} \ . \tag{2.10}$$

With these definitions, it follows that $\pi\tau = \tau\pi$.

2.2 Masslessness and hyperlightlikeness

From the commutation rules (2.3) it follows that there are no singular vectors if e_0 is large enough at fixed $\mathbf{m}_0$, in which case $\mathfrak{D}(e_0, \mathbf{m}_0) = \mathfrak{V}(e_0, \mathbf{m}_0)$ is unitary and is referred to as a massive representation. Lowering e_0 while keeping $\mathbf{m}_0$ fixed, singular vectors appear for the first time at a critical value of e_0 [31]. In case $m_0^1 \geq 1$, the singular vector corresponds

to a field-theoretic gauge artifact [32], and one may refer to the critical $\mathfrak{D}(e_0, \mathbf{m}_0)$ as a massless representation. The simplest case is $\mathbf{m}_0 = (m_0^1) \equiv (m)$ with $m \geq 1$, i.e. a ground state that is a symmetric rank m tensor. The corresponding unitary lowest-weight space is

$$\mathfrak{D}(m + 2\epsilon_0, (m)) \,, \qquad \epsilon_0 \;=\; \frac{D-3}{2} \,, \tag{2.11}$$

and the gauge modes are generated from the longitudinal singular vector

$$|m + 2e_0 + 1, (m-1)\rangle_{r(m-1)} \;=\; \sum_{t=1}^{D-1} L_t^+ |m + 2\epsilon_0, (m)\rangle_{tr(m-1)} \;, \tag{2.12}$$

where we use the notation $r(m) \equiv r_1 \cdots r_m$.

Interestingly, also scalar and spinor Harish-Chandra modules, i.e. the cases $m = 0, 1/2$, exhibit critical behavior giving rise to the *scalar and spinor singletons*

$$\mathfrak{D}_0 \equiv \mathfrak{D}(\epsilon_0, (0)) \,, \qquad \mathfrak{D}_{1/2} \;\equiv\; \mathfrak{D}(\epsilon_0 + 1/2, (1/2)) \;. \tag{2.13}$$

The singular vectors are now given by

$$L_r^+ L_r^+ |\epsilon_0, (0)\rangle \,, \qquad (\gamma_r)_\alpha{}^\beta L_r^+ |\epsilon_0 + 1/2, (1/2)\rangle_\beta \;, \tag{2.14}$$

where $(\gamma_r)_\alpha{}^\beta$ are Dirac matrices (squaring the Dirac operator one finds that $L_r^+ L_r^+ \simeq 0$ also for the spinor). The singletons therefore consist of single discrete lines $\{n+\epsilon_0, (n)+\mathbf{m}_0\}_{n=0}^\infty$ in the non-compact weight space. In the Harish-Chandra module, the singular vectors are compact, or Bogoliubov-transformed, versions of the $(D-1)$-dimensional equations of motion of conformal scalar and spinor fields [33]. There are also unitary singleton-like representations with $m \geq 1$, corresponding to discrete subcritical values of e_0, but we shall not be interested in them here.

Alternatively, the quantum-mechanical equation of motion of the singleton can be written directly in terms of the angular momenta M_{AB} on a manifestly $(D+1)$-dimensionally covariant form as the following *hyperlight-likeness condition*,[5]

$$\langle \Psi | V_{AB} | \Psi' \rangle \;=\; 0 \;, \tag{2.15}$$

where V_{AB} is the traceless operator

$$V_{AB} \;=\; \frac{1}{2} M_{(A}{}^C M_{B)C} - \frac{\eta_{AB}}{D+1} C_2 \;, \qquad C_2 \;=\; \frac{1}{2} M^{AB} M_{AB} \;. \tag{2.16}$$

In a lowest or highest-weight space, this is equivalent to that the ground state $|\Omega\rangle$ obeys

$$V_{rs} |\Omega^\pm\rangle \;=\; 0 \;. \tag{2.17}$$

[5]Realizing the angular momenta as $M_{AB} = X_A P_B - X_B P_A$ where X^A and P^B are the coordinates and momenta of the conformal particle in $\mathbb{R}^{2,D-1}$, one can show that V_{AB} is proportional to the Sp(2) generators X^2, P^2 and $\{X^A, P_A\}$, so that $V_{AB} \simeq 0$ comprises the light-likeness condition $P^2 \simeq 0$ as well as the hyper-cone condition condition $X^2 \simeq 0$. The hyperlight-likeness condition forces the D-dimensional space-time energy-momentum into rotational motion rather than ordinary geodesic, e.g. light-like, motion.

In the case of scalar and spinor lowest-weight spaces in $D \geq 4$, one can show that the hyperlight-likeness condition holds only for scalar singletons in $D \geq 4$ and the spinor singleton in $D = 4$, i.e.[6]

$$\mathfrak{D}_0^{\pm}: \quad \text{hyperlight-like for all } D\,, \tag{2.18}$$

$$\mathfrak{D}_{1/2}^{\pm}: \quad \text{hyperlight-like iff } D = 3, 4\ . \tag{2.19}$$

We note that in the singleton Harish-Chandra module $V_{AB}|\chi\rangle$ are null states for arbitrary $|\chi\rangle$. In particular, the $(D-1)$-dimensional mass-shell condition is recovered from $V^{++} = \frac{1}{2}L_r^+ L_r^+$, where $X^{\pm} = X_{0'} \mp iX_0$. Moreover, acting on $|\epsilon_0 + n, (m)\rangle \in \mathfrak{V}(\epsilon_0, (0))$ with $m = n - 2k$, $k = 1, 2, \ldots$ (i.e. containing k factors of $L_r^+ L_r^+$) with V_r^+ and decomposing into symmetric traceless and trace parts, yields

$$V_{\{r}^+|\epsilon_0 + n, (m)\rangle_{r(m)\}} = ikL_{\{r}^+|\epsilon_0 + n, (m)\rangle_{r(m)\}}\ , \tag{2.20}$$

$$V_s^+|\epsilon_0 + n, (m)\rangle_{sr(m-1)} = \frac{i}{2}L_r^+|\epsilon_0 + n, (m)\rangle_{rs(m-1)}\ , \tag{2.21}$$

where $\{r(m)\}$ denotes the traceless part of $r(m)$. Each of these equations separately implies that all states in the Harish-Chandra module with $k > 0$ are null states.

2.3 Compositeness and higher-spin algebra

The fundamental nature of the singletons was first exhibited in $D = 4$ by Flato and Frønsdal in [25], where they derived the following decomposition under $\mathfrak{so}(2,3)$ of the tensor product of two scalar singletons,

$$\mathfrak{D}_0 \otimes \mathfrak{D}_0 = \bigoplus_{m=0,1,2,\ldots} \mathfrak{D}(m+1, m)\ , \tag{2.22}$$

$$\mathfrak{D}_0 \otimes \mathfrak{D}_{1/2} = \bigoplus_{2m=1,3,\ldots} \mathfrak{D}(m+1, m)\ , \tag{2.23}$$

$$\mathfrak{D}_{1/2} \otimes \mathfrak{D}_{1/2} = \mathfrak{D}(2,0) \oplus \bigoplus_{m=1,2,\ldots} \mathfrak{D}(m+1, m)\,, \tag{2.24}$$

where we note that all representations on the right-hand sides are massless. Their result generalizes straightforwardly to two-singleton composites D dimensions [34 – 38, 15]

$$\mathfrak{D}_0 \otimes \mathfrak{D}_0 = \bigoplus_{m=0,1,2,\ldots} \mathfrak{D}(m + 2\epsilon_0, (m))\ . \tag{2.25}$$

For this reason, one refers to the symmetric traceless rank $m \geq 0$ tensors in D dimensions and the pseudo-scalar $\mathfrak{D}(2,0)$ in $D = 4$ as composite massless.[7] The two-singleton compos-

[6] One can also show that the scalar and spinor singletons in $D = 3$ are hyperlight-like.

[7] In $D = 4$ the composite massless representations admit extensions to the conformal group whose restrictions to the Poincaré group are ordinary massless particles with helicity $\pm m$. These can be compared with the conformally coupled scalar in D dimensions, that has $e_0 = (D - 1 \pm 1)/2$, which can equal the composite value $e_0 = D - 3$ only in $D = 4$ and $D = 6$, and that of the composite pseudo-scalar value $e_0 = D - 2$ only in $D = 2$ and $D = 4$.

ites can be decomposed under $\mathcal{S}_2$ into symmetric and anti-symmetric Young projections[8]

$$[\mathfrak{D}_0 \otimes \mathfrak{D}_0]_{\rm S} = \bigoplus_{m=0,2,4,\ldots} \mathfrak{D}(m+2\epsilon_0,(m))\,, \tag{2.26}$$

$$[\mathfrak{D}_0 \otimes \mathfrak{D}_0]_{\rm AS} = \bigoplus_{m=1,3,5,\ldots} \mathfrak{D}(m+2\epsilon_0,(m))\ . \tag{2.27}$$

Adhering to the nomenclature introduced in [41], we shall refer to Young-projected multi-singleton composites as *multipletons*, and in particular to the massless cases as symmetric and anti-symmetric doubletons.[9] In general, the tensor product of $P \geq 2$ singletons decomposes under $\mathcal{S}_P$ into $P!$ P-tupletons. For $P > 2$ these decompose under $\mathfrak{so}(2,D-1)$ into massive representations, although as far as we know the explicit form of this decomposition has not been given in the literature.

The scalar singleton $\mathfrak{D}_0^{\pm}$ is the fundamental representation of the minimal bosonic higher-spin algebra $\mathfrak{hs}_0(2,D-1)$. This algebra is an infinite-dimensional Lie-algebra extension of $\mathfrak{so}(2,D-1)$, defined by factoring out the annihilator of $\mathfrak{D}_0^{\pm}$ from the $Env(\mathfrak{so}(2,D-1))$ [38, 43, 15].[10] The algebra $\mathfrak{hs}_0(2,D-1)$ acts transitively on $\mathfrak{D}_0^{\pm}$ (and $\mathfrak{D}_{1/2}^{\pm}$ in $D=4$) and irreducibly on the multipletons. The irreducibility of the multipletons, i.e. the absence of non-trivial $\mathfrak{hs}_0(2,D-1)$-invariant tensors in $(\mathfrak{D}_0^{\pm})^{\otimes P}$, is a consequence of the fact that $\mathfrak{hs}_0(2,D-1)$ contains $\mathfrak{su}(k)$ subalgebras for unbounded values of k. These subalgebras are obtained by consistently truncating to generators that have non-vanishing matrix elements only between singleton states with excitation energies ϵ_0+n for $n=0,1,\ldots,N$ (k can be made arbitrarily large by taking N sufficiently large). As an aside, for later reference, we note that both $(\mathfrak{hs}_0(2,D-1),\mathfrak{D}_0^{+})$ and $(\mathfrak{hs}_0(2,D-1),\mathfrak{D}_0^{-})$ are formal $N\to\infty$ limits of $(\mathfrak{su}(N),N)$. Thus, apart from the standard $\mathfrak{hs}_0(2,D-1)$-invariant inner product $\langle\cdot|\cdot\rangle : \mathfrak{D}_0^{\pm}\otimes\mathfrak{D}_0^{\pm}\to\mathbb{C}$, there also exists an $\mathfrak{hs}_0(2,D-1)$-invariant *bilinear* inner product $(\cdot,\cdot) : \mathfrak{D}_0^{\pm}\otimes\mathfrak{D}_0^{\mp}\to\mathbb{C}$ defined by

$$\big(\lambda|\Psi\rangle,\lambda'|\Psi'\rangle\big) = \lambda\lambda'\,\big(|\Psi\rangle,|\Psi'\rangle\big)\ , \tag{2.28}$$

$$\big(|\Psi\rangle,M_{AB}|\Psi'\rangle\big) = \big(-M_{AB}|\Psi\rangle,|\Psi'\rangle\big)\ , \tag{2.29}$$

$$\big(|\epsilon_0,(0)\rangle^-,|\epsilon_0,(0)\rangle^+\big) = \big(|\epsilon_0,(0)\rangle^+,|\epsilon_0,(0)\rangle^-\big) = 1\ . \tag{2.30}$$

The higher-spin algebra, by its construction, contains an infinite-dimensional Cartan subalgebra, so that its representation theory falls outside that of Lie algebras with finite-dimensional Cartan matrices. On the other hand, from a physical point-of-view, the higher-spin algebra appears to be too small to unify interactions that mix multipletons with different values of P (with the exception of (classically consistent truncations to) self-interactions in the massless $P=2$ sector). As found in [15], multipletons arise in a

[8] It is intriguing that $[\mathfrak{D}_0\otimes\mathfrak{D}_0]_{\rm S}$ agrees with $[\mathfrak{D}_{1/2}\otimes\mathfrak{D}_{1/2}]_{\rm AS}$ except in the scalar sector, where the former contains the parity-invariant scalar $\mathfrak{D}(1,0)$ and the latter the pseudo-scalar $\mathfrak{D}(2,0)$. This points via holography to some form of bose-fermi correspondence in three-dimensional conformal field theory [39, 40].

[9] The doubletons of $\mathfrak{so}(2,3)$, which are massless conformal tensors in four dimensions, are singletons of $\mathfrak{so}(2,4)\simeq\mathfrak{su}(2,2)$. This particular notion of doubletons was first introduced in [42].

[10] The minimal higher-spin algebra is denoted by $ho(1|2;[2,D-1])$ in [38].

discretized description of tensionless extended objects in anti-de Sitter spacetime. It was further argued that the continuum limit leads to a non-compact WZW model, wherein the singletons arise as twist-fields in the R sector,[11] the tensoring is lifted to fusion, and the role of the higher-spin algebra is replaced by an affine extension of $\mathfrak{so}(2, D-1)$.

3. On subcritical $\widehat{\mathfrak{so}}(2, D-1)$ WZW models

In this section we propose an $\widehat{\mathfrak{so}}(2, D-1)$ WZW model at the subcritical level $-(D-3)/2$, which we claim accommodates the scalar singleton as well as all its composites. Here, our approach is purely algebraic, based on using spectral flow to solve the current-algebra version of the hyperlight-likeness condition (2.15). The construction will then be explored in more detail in the case of $D = 4$ in the coming sections using symplectic bosons and free-field realizations. The affine representation spaces contain many unphysical states over and above the desired singletons and multipletons, which one may think of as generalized ground states. Whether a unitary model can be extracted by some form of gauging is still an open issue, and we refer to [20] for more details.

3.1 Affine hyperlightlikeness

In what follows we shall focus on the holomorphic sector of the WZW model. The underlying $\widehat{\mathfrak{so}}(2, D-1)_k$ currents obey the operator product expansion

$$M_{AB}(z)M_{CD}(w) = \frac{2k\eta_{AC}\eta_{BC}}{(z-w)^2} + \frac{4i\eta_{BC}M_{AD}(w)}{z-w} + \text{finite}\,, \tag{3.1}$$

where the level k is chosen such that the normal-ordered field[12]

$$V_{AB}(z) = \frac{1}{2}\left(M_{(A}{}^{C}M_{B)C}\right)(z) - \text{trace}\,, \tag{3.2}$$

is a singular vector in the NS sector. In other words, k is fixed such that $V_{AB}(z)$ is a Kac-Moody primary obeying

$$M_{AB}(z)V_{CD}(w) = \frac{4i\eta_{BC}V_{AD}(z)}{z-w} + \text{finite}\,. \tag{3.3}$$

Since V_{AB} is an $\mathfrak{so}(2, D-1)$ tensor with highest weight (2) and canonical conformal weight 2, the Sugawara construction implies that V_{AB} can be a KM primary iff

$$h[V_{AB}] \equiv \frac{C_2[\mathfrak{so}(2, D-1)|(2)]}{2(k+h^\vee)} = \frac{2(2+D-1)}{2(k+D-1)} = 2\,, \tag{3.4}$$

from which we read off the *subcritical level*

$$k = -\epsilon_0\,. \tag{3.5}$$

[11] In [15] we used the term spin-field, but here we shall instead use the more commonly used term twist-field.

[12] The normal ordering operation is defined by $(AB)(z) = \oint_z \frac{dx}{2\pi i}\frac{A(x)B(z)}{x-z}$ (for example, see [44]).

Indeed, for this value the double contractions in $M_{AB}(z)V_{CD}(w)$ cancel, and one is left with (3.3). The Sugawara stress tensor becomes

$$T(z) = \frac{1}{2(D+1)}(M^{AB}M_{AB})(z) \tag{3.6}$$

with central charge $c = -D(D-3)/2$. The hermitian conjugation is given by $X^\dagger = ((X)^*)^\tau$ where $*$ is an anti-linear automorphism $*$ and τ a linear anti-automorphism (BPZ conjugation), defined by

$$(M_{AB}(z))^* = -M_{AB}(z)\,, \qquad (M_{AB}(z))^\tau \;=\; -\frac{1}{z^2}M_{AB}\left(z^{-1}\right)\,, \tag{3.7}$$

resulting in

$$(M_{AB,n})^* = -M_{AB,n}\,, \qquad (M_{AB,n})^\tau \;=\; -M_{AB,-n}\,, \qquad (M_{AB,n})^\dagger \;=\; M_{AB,-n}\,. \tag{3.8}$$

Acting on the 0-modes, the BPZ conjugation reduces to the anti-automorphism τ defined in (2.10). The lift of the $\mathfrak{so}(2,D-1)$ automorphism π given in (2.9) to the affine case is given by

$$\pi(M_{ab,n}) = M_{ab,n}\,, \qquad \pi(P_{a,n}) \;=\; -P_{a,n}\,. \tag{3.9}$$

Turning to the primary fields of the WZW model, the decoupling of the singular vacuum vector at the level of three-point functions forces the primaries to obey an *equation of motion* [16, 21], which is a necessary condition that fixes the admissible representations but not their multiplicities (a complete determination of the spectrum requires further input from demanding closure of the operator product expansion, crossing symmetry and modular invariance). Denoting the primary fields and states by $V_\Lambda(z)$ and $|\Lambda\rangle = V_\Lambda(0)|0\rangle$, and the corresponding KM Verma modules by $\widehat{\mathfrak{V}}(\Lambda)$, the decoupling amounts to

$$\langle\Psi|V_{AB,n}|\Psi'\rangle = 0 \quad \text{for all } n\in\mathbb{Z} \text{ and } |\Psi\rangle,\ |\Psi'\rangle \in \widehat{\mathfrak{V}}(\Lambda)\,, \tag{3.10}$$

where the inner products are defined using the hermitian conjugation (3.8). The decoupling (3.10) is equivalent to $\langle\Lambda|V_{AB,n}|\Lambda\rangle = 0$ for all n. Assuming that $\langle\Lambda|$ has a fixed L_0 eigenvalue, this is the same as $\langle\Lambda|V_{AB,0}|\Lambda\rangle = 0$. Finally, assuming that $\langle\Lambda|$ is a $\mathfrak{so}(2,D-1)$ ground state, one finds

$$V_{rs,0}|\Lambda\rangle = 0\,, \tag{3.11}$$

which is the affine version of (2.17).

In general, the primary field may be *twisted* in the sense that $M_{AB,n}|\Lambda\rangle$ vanishes for some AB and $n<0$ and does not vanish for some AB and $n>0$. In our case, the relevant twisting is described using the compact basis, *viz.*

$$[L^-_{r,m}, L^+_{s,n}] = 2iM_{rs,m+n} + 2\delta_{rs}E_{m+n} + 2\epsilon_0 m\delta_{rs}\delta_{m+n,0}\,, \tag{3.12}$$

$$[E_m, E_n] = -\epsilon_0 m\delta_{m+n,0}\,, \tag{3.13}$$

$$[M_{rs,m}, M_{tu,n}] = 4i\delta_{st}M_{ru,m+n} - 2\epsilon_0 m\delta_{rt}\delta_{su}\delta_{m+n,0}\,. \tag{3.14}$$

Given an integer P, a *P-twisted primary field*[13] $V_{[P];\Lambda}(z)$ is by definition taken to obey

$$L^{\pm}_{r,n}|[P];\Lambda\rangle \;=\; 0 \quad \text{for } n \geq \pm P + 1\,, \tag{3.15}$$

$$M_{rs,n}|[P];\Lambda\rangle \;=\; E_n|[P];\Lambda\rangle \;=\; 0 \quad \text{for } n \geq 1\,, \tag{3.16}$$

and to be a ground state with lowest weight labels $(e_0^{[P]}, \mathbf{m}_0^{[P]})$ of a Harish-Chandra module of the $\mathfrak{so}(2, D-1)$ algebra with generators

$$L_r^{[P]\pm} = L^{\pm}_{r,\pm P}\,, \qquad M_{rs}^{[P]} \;=\; M_{rs,0}\,, \qquad E^{[P]} \;=\; E_0 - P\epsilon_0\ . \tag{3.17}$$

We note that from $(E^{[P]} - e_0^{[P]})|[P],\Lambda\rangle = 0$ it follows that

$$e_0 \;=\; e_0^{[P]} + P\epsilon_0\,, \qquad \mathbf{m}_0 \;=\; \mathbf{m}_0^{[P]}\ . \tag{3.18}$$

We shall label the P-twisted ground states by

$$\Lambda \;=\; (h; e_0, \mathbf{m}_0)\,, \tag{3.19}$$

where h is the conformal weight. The action of the KM creation operators on $|[P]; h; e_0, \mathbf{m}_0\rangle$ generates Verma modules $\widehat{\mathfrak{V}}_{[P];h}(e_0, \mathbf{m}_0)$ containing null submodules $\widehat{\mathfrak{N}}_{[P];h}(e_0, \mathbf{m}_0)$ generated by P-twisted singular vectors (containing at least all excitations generated by the modes of V_{AB}). Factoring out the null states yields the twisted lowest weight spaces

$$\widehat{\mathfrak{D}}_{[P];h}(e_0; \mathbf{m}_0) = \frac{\widehat{\mathfrak{V}}_{[P];h}(e_0; \mathbf{m}_0)}{\widehat{\mathfrak{N}}_{[P];h}(e_0; \mathbf{m}_0)}\ . \tag{3.20}$$

We note that the standard definition of a KM weight space is recovered for $P = 0$. For $P = \pm 1$, one additional singular vector has been factored out from the Verma module, namely $L^{\mp}_{r,-1}|[\pm 1];\Lambda\rangle$.

3.2 Twisted primary scalars and multipletons

Let us consider the special case of a P-twisted primary field $V_{[P]}(z)$ that is a *singlet* under $\mathfrak{so}(2, D-1)_{[P]}$, i.e.

$$L^{\pm}_{r,n}|[P]\rangle \;=\; 0\,, \qquad n \geq \pm P\ , \tag{3.21}$$

$$(E_n - \delta_{n0} P\epsilon_0)|[P]\rangle \;=\; 0\ , \qquad M_{rs,n}|[P]\rangle \;=\; 0\,, \qquad n \geq 0\ . \tag{3.22}$$

These states are also conformal primaries,

$$(L_n - h_{[P]}\delta_{n0})|[P]\rangle \;=\; 0\,, \qquad h_{[P]} \;=\; -\frac{P^2\epsilon_0}{2}\ , \tag{3.23}$$

where $n \geq 0$ and L_n are the Virasoro generators taken from the Sugawara stress tensor (3.6). Since $|[P]\rangle$ is an $\mathrm{SO}(D-1)$ scalar, it follows that

$$V_{rs,0}|[P]\rangle \;=\; \frac{1}{D-1}\delta_{rs} V_{tt,0}|[P]\rangle\ . \tag{3.24}$$

[13] Primary fields of this type are also referred to in the literature (see e.g. [4, 45]) as spectrally flowed primary fields.

By explicit calculations one can then go on to show that $V_{rr,0}|[P]\rangle = 0$, i.e. the twisted-primary singlets defined above obey the equations of motion (3.11), i.e.

$$V_{rs,0}|[P]\rangle = 0 \ . \tag{3.25}$$

The scalar twisted lowest-weight spaces, that we shall denote by

$$\widehat{\mathfrak{D}}_{[P]} = \widehat{\mathfrak{D}}_{[P],-\frac{P^2\epsilon_0}{2}}(P\epsilon_0,(0))\,, \tag{3.26}$$

contain negative norm states, which should be removed carefully by imposing suitable gauge conditions [7, 20]. In this paper we shall not enter this important discussion. Instead, let us highlight the interesting, potentially physical, subspace $\mathfrak{D}_{[P]} \subset \widehat{\mathfrak{D}}_{[P]}$ given by

$$\mathfrak{D}_{[P]} = \bigoplus_{e_0,\mathbf{m}_0} \mathfrak{D}_{[P];h}(e_0,\mathbf{m}_0)\,, \tag{3.27}$$

where $\mathfrak{D}_{[P];h}(e_0,\mathbf{m}_0)$ are defined to be the (untwisted) $\mathfrak{so}(2,D-1)$ weight spaces generated by acting with $L^\sigma_{r,0}$, with $\sigma = \mathrm{sign}(P)$, on ground states of the form

$$|[P];h;e_0,\mathbf{m}_0\rangle^\sigma = \left(\prod_{r,|n|\le|P|-1} L^\sigma_{r,n}\right)|[P]\rangle + |[P];\Psi\rangle\,, \tag{3.28}$$

where the state $|[P];\Psi\rangle$ contains at least one excitation by one of the compact subalgebra generators $\{E_{-n}, M_{rs,-n}\}_{n=0}^{|P|-1}$, and is determined by

$$L^{-\sigma}_{r,0}|[P];h;e_0,\mathbf{m}_0\rangle^\sigma = 0 \ . \tag{3.29}$$

We note that the weight spaces in $\mathfrak{D}_{[P]}$ are of lowest-weight type for $P>0$ and highest-weight type for $P<0$, and we shall therefore drop the superscript σ on the states. In particular,

$$|[P]\rangle = |[P];-\frac{P^2\epsilon_0}{2};2\sigma\epsilon_0,(0)\rangle \tag{3.30}$$

are ground states of scalar weight spaces. The state $|[0]\rangle$ is the NS vacuum, so that

$$\mathfrak{D}_{[0]} = \mathfrak{D}_{[0]}(0,0) \ . \tag{3.31}$$

For $P=\pm1$, we identify $|[\pm1]\rangle$ as the ground states of the scalar singleton, $\mathfrak{D}^+ = \mathfrak{D}_{[1];-\epsilon_0/2}(\epsilon_0,(0))$, and anti-singleton, $\mathfrak{D}^- = \mathfrak{D}_{[-1];-\epsilon_0/2}(-\epsilon_0,(0))$, and hence

$$\mathfrak{D}_{[\pm1]} \simeq \mathfrak{D}_0^\pm \ . \tag{3.32}$$

For $P=\pm2$, we identify $|[2]\rangle$ and $|[-2]\rangle$ as the ground states of the composite massless scalar and anti-scalar, respectively. Acting on $|[\pm2]\rangle$ with traceless strings of $L^+_{r,1}$ generators yields the *composite massless higher-spin ground states* ($m=0,1,2,3,\dots$)

$$|[\pm2];-2\epsilon_0-m;\pm(m+2\epsilon_0),(m)\rangle_{\{r(m)\}} \ = \ L^\pm_{\{r_1,1}\cdots L^\pm_{r_m\},1}|[\pm2]\rangle \ . \tag{3.33}$$

We note that the trace parts sit in $\widehat{\mathfrak{N}}_{[\pm 2];-2\epsilon_0-m}(\pm(m+2\epsilon_0),(m))$. Similarly, traceless strings of $L^{\sigma}_{r,-1}$ generators dressed up with suitable correction terms involving subalgebra generators provide massless ground states of the form $|[\pm 2];-2\epsilon_0+m;m+2\epsilon_0,(m)\rangle$. For example, the photon ground state with conformal weight $-2\epsilon_0+1$ is given by

$$|[2];-2\epsilon_0+1;1+2\epsilon_0,(1)\rangle_r \;=\; \left(L^{+}_{-1,r}+\frac{i}{1+2\epsilon_0}M_{rs,-1}L^{+}_{s,0}\right)|[2]\rangle \;. \tag{3.34}$$

Similar constructions can be given for higher spin and for $P=-2$. Thus

$$\mathfrak{D}_{[\pm 2]} \;\supset\; \mathfrak{D}_{[\pm 2];-2\epsilon_0}(\pm 2\epsilon_0,(0)) \;\oplus\; \mathfrak{D}'_{[\pm 2];+} \;\oplus\; \mathfrak{D}'_{[\pm 2];-} \;, \tag{3.35}$$

where we have defined

$$\mathfrak{D}'_{[\pm 2];-} \;=\; \bigoplus_{m=1,2,\ldots} \mathfrak{D}_{[\pm 2];-2\epsilon_0-m}(\pm(m+2\epsilon_0),(m)) \,, \tag{3.36}$$

$$\mathfrak{D}'_{[\pm 2];+} \;=\; \bigoplus_{m=1,2,\ldots} \mathfrak{D}_{[\pm 2];-2\epsilon_0+m}(\pm(m+2\epsilon_0),(m)) \;. \tag{3.37}$$

From (2.25) it follows that

$$\mathfrak{D}_{[\pm 2];-2\epsilon_0}(\pm 2\epsilon_0,(0)) \oplus \mathfrak{D}'_{[\pm 2];-} \;\simeq\; \mathfrak{D}^{\pm}_0 \otimes \mathfrak{D}^{\pm}_0 \;. \tag{3.38}$$

We expect the above pattern to generalize, so that

$$\mathfrak{D}_{[P]} \;\supset\; \mathfrak{D}_{[P];-} \;\simeq\; (\mathfrak{D}^{\sigma}_0)^{\otimes |P|} \,, \tag{3.39}$$

where $\mathfrak{D}_{[P];-}$ is defined to be the space of states in $\mathfrak{D}_{[P]}$ with minimal L_0-eigenvalue for fixed $\mathfrak{so}(2,D-1)$ quantum numbers.

The above analysis suggests that the scalar twisted primary states $|[P]\rangle$ correspond to scalar twisted primary fields $V_{[P]}$ obeying the simple fusion rule $V_{[P]} \times V_{[P']} = V_{[P+P']}$.

3.3 Spectrum-generating flow and fusion

Spectral flow is an operation which shifts the modes of affine generators such that the spectrally-flowed algebra is isomorphic to the original one. It is known that spectral flow in WZW models based on affine Lie algebras with infinite-dimensional zero-mode representations connects an infinite set of sectors which are all required for the consistency of the model, see *e.g.* [46, 4, 21, 45]. In our case, these sectors are labeled by the integer P, containing as "zero modes" states in the $|P|$th tensor product of singletons ($P>0$) or antisingletons ($P<0$).

The possible spectra of scalar twisted primaries is restricted by a delicate interplay between fusion and spectral flow. At the level of the fusion rules, which we denote by $\times$ and whose entries are affine representation labels Λ, the spectral flow operation, *viz.*

$$\Omega_P[V_\Lambda] = V_{\Omega_P(\Lambda)} \,, \qquad P \in \mathbb{Z} \,, \tag{3.40}$$

can be composed as follows [46]

$$\Omega_P \circ \Omega_{P'} \;=\; \Omega_{P+P'} \;. \tag{3.41}$$

By using the commutativity of fusion rules, one can also show that

$$\Omega_P[V_\Lambda \times V_{\Lambda'}] = \Omega_P[V_\Lambda] \times V_{\Lambda'} \;=\; V_\Lambda \times \Omega_P[V_{\Lambda'}] \; . \tag{3.42}$$

We note that as a consequence $\Omega_{P+P'}[V_\Lambda \times V_{\Lambda'}] = \Omega_P[V_\Lambda] \times \Omega_{P'}[V_{\Lambda'}]$. At the level of the operator product algebra the distribution of spectral flows induces automorphisms,

$$\Omega_P[A(z)B(w)] = (\Omega_P A \Omega_P^{-1})(z)(\Omega_P[B])(w) \; . \tag{3.43}$$

The spectrally flowed currents are defined by

$$\begin{aligned} (\Omega_P L_r^\pm \Omega_P^{-1})(z) &= z^{\pm P} L_r^\pm(z) \, , \\ (\Omega_P M_{rs} \Omega_P^{-1})(z) &= M_{rs}(z) \, , \end{aligned} \tag{3.44}$$

$$(\Omega_P E \Omega_P^{-1})(z) = E(z) - P\epsilon_0 z^{-1} \, , \tag{3.45}$$

or in terms of the charges

$$\Omega_P L^\pm_{r,n} \Omega_P^{-1} = L^\pm_{r,n\pm P} \, , \qquad \Omega_P M_{rs,n} \Omega_P^{-1} \;=\; M_{rs,n} \, , \tag{3.46}$$

$$\Omega_P E_n \Omega_P^{-1} = E_n - P\epsilon_0 \delta_{n,0} \; . \tag{3.47}$$

The shift in the energy is compatible with the central extension in (3.12). The spectral flow of the Sugawara stress tensor (3.6) is by definition given by

$$(\Omega_P T \Omega_P^{-1})(z) = \oint_z \frac{dx}{2\pi i (x-z)} (\Omega_P M^{AB} \Omega_P^{-1})(x) (\Omega_P M_{AB} \Omega_P^{-1})(z) \; . \tag{3.48}$$

Expanding the operator product and evaluating the residue at $x = z$ one finds

$$(\Omega_P T \Omega_P^{-1})(z) = T(z) + PE(z) - \frac{P^2 \epsilon_0}{2} z^{-2} \; , \tag{3.49}$$

or, at the level of Virasoro generators,

$$\Omega_P L_n \Omega_P^{-1} = L_n + PE_n - \frac{P^2 \epsilon_0}{2} \delta_{n,0} \; , \tag{3.50}$$

where $\Omega_P L_n \Omega_P^{-1}$ indeed obeys the same Virasoro algebra as L_n. The mixing between $T(z)$ and the AdS-energy current $E(z)$ can also be expressed as

$$E(z) = (\Omega_P E \Omega_P^{-1})(z) + P\epsilon_0 z^{-1} \, , \tag{3.51}$$

$$T(z) = (\Omega_P T \Omega_P^{-1})(z) - P(\Omega_P E \Omega_P^{-1})(z) z^{-1} - \frac{P^2 \epsilon_0}{2} z^{-2} \; . \tag{3.52}$$

A family of $\mathfrak{so}(2, D-1)_{[P]}$-singlet P-twisted primary fields, obeying (3.21) and (3.22), can now be constructed by applying spectral flow to the identity,

$$V_{[P]} = \Omega_P[\mathbb{I}] \, , \qquad P \in \mathbb{Z} \; . \tag{3.53}$$

By construction these fields obey the hyperlight-likeness condition (3.11), which one can also verify directly by calculating

$$\Omega_P V_{rs,0} \Omega_P^{-1} = V_{rs,0}\,, \tag{3.54}$$

which implies $V_{rs,0}|[P]\rangle = \Omega_P[V_{rs,0}|0\rangle] = \Omega_P[0] = 0$. We note that $\Omega_P E_n \Omega_P^{-1}|[P]\rangle = \Omega_P L_n \Omega_P^{-1}|[P]\rangle = 0$ for $n \geq 0$ is consistent with (3.47) and (3.50), and that the operators on the left and right-hand sides of (3.51) and (3.52) are normal-ordered with respect to the 0-twisted and P-twisted ground states, respectively. The various conjugates of the twisted primary states are given by

$$|[P]\rangle^* \;=\; |[-P]\rangle\,, \qquad |[P]\rangle^\tau \;=\; \langle[P]|\,, \qquad \pi(|[P]\rangle \;=\; |[-P]\rangle\,, \tag{3.55}$$

implying the hermitian conjugates

$$|[P]\rangle^\dagger \;\equiv\; (|[P]\rangle^*)^\tau \;=\; \langle[-P]|\;. \tag{3.56}$$

We can thus define an inner product by declaring

$$\langle[P']|[P]\rangle = \delta_{P+P',0}\,, \qquad P, P' \in \mathbb{Z}\;. \tag{3.57}$$

By definition $\langle[P]|M_{AB,n} = ((M_{AB,n})^\dagger|[-P]\rangle)^\dagger$, which implies that the above definition is consistent with the assignment of E_0 eigenvalues. We note that there also exists a bilinear inner product $(\cdot,\cdot)$, induced by the two-point functions, *viz.*

$$\langle 0|O_\Psi(z)O_{\Psi'}(w)|0\rangle \;=\; \frac{(|\Psi\rangle, |\Psi'\rangle)}{(z-w)^{2h(\Psi)}}\,, \tag{3.58}$$

where $O_\Psi(z)$ denotes the vertex operator corresponding to the state $|\Psi\rangle$. This bilinear inner product reduces to (2.30) upon restricting to the set of $[\pm]$-states generated by the 0-modes.

The composition rule (3.41) and distribution rule (3.42) imply the simple fusion rule[14]

$$V_{[P]} \times V_{[P']} = V_{[P+P']}\;. \tag{3.59}$$

The analysis so far suggests an $\widehat{\mathfrak{so}}(2, D-1)_{-\epsilon_0}$ model with spectrum given by

$$\widehat{\mathfrak{D}} = \bigoplus_{P\in\mathbb{Z}} \widehat{\mathfrak{D}}_{[P];-\frac{P^2\epsilon_0}{4}}(P\epsilon_0, (0))\;. \tag{3.60}$$

Clearly, the closure of the fusion rules is only a necessary criterion, and we leave a more detailed study of this model, e.g. the properties under modular transformations and locality of the operator product expansion, to future work. In fact, already in the case of $D = 4$, we observe that the above assertion does not take into account the hyperlight-likeness of the spinor singleton, *cf.* (2.19). As we shall show next, the affine extension of this representation can be added and spectrally flowed leading to an *extended* 4D model with additional tensor-spinors.

[14]In the case of WZW models based on affine algebras with infinite-dimensional zero-mode representations, one has to pay extra attention to the fusion rules. For instance, the Verlinde formula does not necessarily apply, *cf.* [21] and references therein.

JHEP02(2007)097

3.4 Twisted primary spinors in $D = 4$

The case of $D = 4$ is special in two respects. First, there is a massless pseudo-scalar in $\mathfrak{D}_{[\pm 2]}$, which has no analog for $D > 4$. It is given by

$$|[\pm 2]; -1; \pm 2, 0\rangle = \epsilon^{rst} M_{rs,-1} L^{\pm}_{t,1} |[\pm 2]\rangle \ . \tag{3.61}$$

Thus, in $D = 4$

$$\mathfrak{D}_{[\pm 2]} \supset \mathfrak{D}_{[\pm 2];-1}(\pm 1, 0) \ \oplus \ \mathfrak{D}_{[\pm 2];-1}(\pm 2, 0) \ \oplus \ \mathfrak{D}'_{[\pm 2];+} \ \oplus \ \mathfrak{D}'_{[\pm 2];-} \tag{3.62}$$

$$\simeq (\mathfrak{D}^{\pm}_{0})^{\otimes 2} \oplus (\mathfrak{D}^{\pm}_{\frac{1}{2}})^{\otimes 2} \,, \tag{3.63}$$

where we have used (2.22) and (2.24). We note that the massless pseudo-scalar is an $\widehat{\mathfrak{sp}}(4)_{-1/2}$ descendant of the massless scalar, while these spaces belong to distinct irreps of the higher-spin algebra $\mathfrak{hs}(4)$, see e.g. [47, 40].

Second, the spinor singleton is hyperlight-like, as we saw in the previous section. The affine extension of the spinor singleton belongs to a 1-twisted sector that is related by spectral flow to the fundamental spinor of $\mathfrak{so}(2,3)$, i.e. the real quartet of $\mathfrak{sp}(4)$ which has lowest energy $-1/2$ and highest energy $1/2$. More generally, the spectral flow gives rise to spinorial P-twisted sectors. To describe these, we decompose the quartet $\mathrm{SO}(2) \times \mathrm{SO}(3)$ into two $\mathrm{SO}(3)$ doublets with energy $\pm 1/2$, that we shall denote by

$$|[0]; \pm\rangle_i = |[0]; \tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}; \pm\rangle_i \ , \tag{3.64}$$

where the quantum numbers indicate that

$$L^{\pm}_{r,n} |[0]; \pm\rangle_i = 0 \ , \qquad n \geq 0 \,, \tag{3.65}$$

$$L^{\mp}_{r,n} |[0]; \pm\rangle_i = \pm \delta_{n0} (\sigma_r)_i{}^j |[0]; \pm\rangle_j \,, \qquad n \geq 0 \,, \tag{3.66}$$

$$\left(E_n \mp \tfrac{1}{2}\delta_{n0}\right) |[0]; \pm\rangle_i = 0 \,, \qquad n \geq 0 \,, \tag{3.67}$$

$$\left(M_{rs,n} |[0]; \pm\rangle_i + \tfrac{i}{2} \delta_{n0} (\sigma_{rs})_i{}^j |[0]; \pm\rangle_j\right) = 0 \,, \qquad n \geq 0 \ . \tag{3.68}$$

The spectral flow operation yields the states

$$|[P]; \pm\rangle_i = |[P]; \tfrac{3-(|P|+1)^2}{4}; \tfrac{P+\sigma}{2}, \tfrac{1}{2}; \pm\rangle_i \ = \ \Omega_P(|[0]; \pm\rangle_i) \,, \tag{3.69}$$

obeying

$$L^{\pm}_{r,n} |[P]; \pm\rangle_i = 0 \,, \qquad n \geq \pm P \,, \tag{3.70}$$

$$L^{\mp}_{r,n} |[0]; \pm\rangle_i = \pm \delta_{n \pm P, 0} (\sigma_r)_i{}^j |[0]; \pm\rangle_j \,, \qquad n \geq \mp P \,, \tag{3.71}$$

$$\left(E_n \mp \tfrac{1}{2}\delta_{n0}(|P|+1)\right) |[P]; \pm\rangle_i = 0 \ , \qquad n \geq 0 \,, \tag{3.72}$$

$$\left(M_{rs,n} |[0]; \pm\rangle_i + \tfrac{i}{2} \delta_{n0} (\sigma_{rs})_i{}^j\right) |[0]; \pm\rangle_j = 0 \,, \qquad n \geq 0 \,, \tag{3.73}$$

$$\left(L_n - \delta_{n0} h_{[P];-\frac{1}{2},\frac{1}{2}}\right) |[P]; \pm\rangle_i = 0 \,, \qquad h_{[P];-\frac{1}{2},\frac{1}{2}} \ = \ \tfrac{3-(|P|+1)^2}{4} \ . \tag{3.74}$$

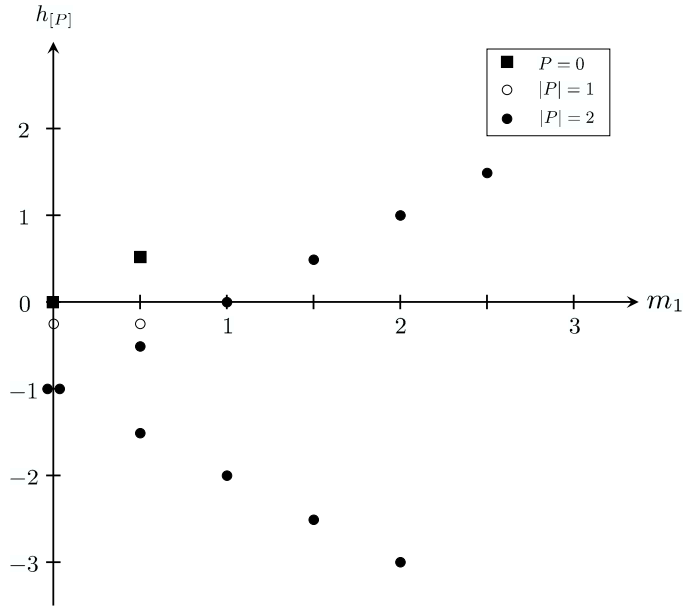

Figure 1: The conformal dimensions $h_{[P]}$ as a function of the spin $m_1 = m$ of the ground states of the $\mathfrak{so}(2,3) \simeq \mathfrak{sp}(4)$ subspaces $\mathfrak{D}_{[P];h}(e_0, \mathbf{m}_0)$ defined in (3.27). Note that there are two scalar ground states having $h = -1$ in each of the $P = \pm 2$ sectors, one of which corresponds to the twisted primary state $|[\pm 2]; -1; \pm 1, 0\rangle$ and the other to the pseudoscalar $|[\pm 2]; -1; \pm 2, 0\rangle$.

The above analysis suggests that if one adds the above spinorial sector to the model in $D = 4$ (with multiplicities equal to 1), then the spinorial contributions to $\mathfrak{D}_{[\pm 2]}$ make this space isomorphic to $(\mathfrak{D}_0^{\pm} \oplus \mathfrak{D}_{1/2}^{\pm})^{\otimes 2}$. The natural generalization would then lead to a

$$\textit{4D extended model:}\ \ \mathfrak{D}_{[P]} \ \simeq\ (\mathfrak{D}_0^{\sigma} \oplus \mathfrak{D}_{\frac{1}{2}}^{\sigma})^{\otimes |P|}\ . \tag{3.75}$$

In figure 1 we summarize the set of ground states specifying the representations $\mathfrak{D}_{[P];h}(e_0, \mathbf{m}_0)$ in the $D = 4$ model for $|P| \leq 2$.

Let us continue with a more detailed analysis in $D = 4$ using a realization in terms of symplectic bosons.

4. The extended $\widehat{\mathfrak{sp}}(4)$ model

In this section we give the realization of the four-dimensional extended model, which has symmetry group $\widehat{\mathfrak{so}}(2,3)_{-1/2} \simeq \widehat{\mathfrak{sp}}(4)_{-1/2}$, in terms of 4 real symplectic bosons forming a Majorana spinor [48, 49],[15] i.e. a quartet of $\mathfrak{sp}(4)$. Indeed, the Sugawara central charge $c = -2$ and conformal weights $h = -1/2$ and $h = -1/4$, respectively, of the fundamental

[15] Also the models in $D = 5$ and $D = 7$ can be realized in terms of symplectic bosons in spinor representations, but these models require additional internal gaugings (U(1) in $D = 5$ and SU(2) in $D = 7$) that are critical for scalar singletons but not for spinor (nor higher-spin) singletons ground states. This provides an explanation why spinor singletons are hyperlight-like in $D = 4$ but not $D > 4$.

spinor and the twist field agree with the canonical values of the system of symplectic bosons (where the R sector corresponds to the twist field). To some extent, this correspondence is analogous to that between the $\widehat{\mathfrak{so}}(4)_1$ model and 4 real fermions. The important difference is that in the fermionic model all conformal weights are bounded from below, so that spectral flow becomes a $\mathbb{Z}_2$-operation, corresponding to moving between weights in the two congruence classes of SO(4), or equivalently, between even and odd momenta in the two-dimensional Euclidean lattice of the free-field realization. In the case of symplectic bosons, the conformal weights are bounded from below by $-P^2/4$ for fixed amount of twist, P. The spectral flow operation now becomes a $\mathbb{Z}$-operation, corresponding to moving between classes of momenta in the four-dimensional lattice of signature $(2,2)$ of the free-field realization, which will be considered in more detail in the next section.

4.1 Realization using symplectic bosons

In order to make contact with the standard oscillator realization of the singletons, it is convenient to work in a $\mathrm{U}(2)$-covariant compact basis where the symplectic bosons are assembled into two doublets $a_i(z)$ and $\bar{a}^i(z)$ $(i = 1, 2)$, obeying[16]

$$a_i(z)\bar{a}^j(w) \sim \frac{\delta_i^j}{z-w} \,. \tag{4.1}$$

In this basis, the $\widehat{\mathfrak{sp}}(4)$ currents are given by

$$\bar{J}^{ij} \,=\, \bar{a}^i\bar{a}^j \,, \qquad K^i{}_j \;=\; (\bar{a}^i a_j) \,, \qquad J_{ij} \;=\; a_i a_j \,, \tag{4.2}$$

and obey

$$J_{ij}(z)\bar{J}^{kl}(w) \sim \frac{4\delta^{(l}_{(j}K^{k)}{}_{i)}(w)}{z-w} - \frac{4k}{(z-w)^2}\delta^k_{(i}\delta^l_{j)} \,, \tag{4.3}$$

$$K^i{}_j(z)\bar{J}^{kl}(w) \sim \frac{\delta^k_j\bar{J}^{il}(w)}{z-w} + \frac{\delta^l_j\bar{J}^{ik}(w)}{z-w} \,, \tag{4.4}$$

$$K^i{}_j(z)J_{kl}(w) \sim -\frac{\delta^i_k J_{jl}(w)}{z-w} - \frac{\delta^i_l J_{jk}(w)}{z-w} \,, \tag{4.5}$$

$$K^i{}_j(z)K^k{}_l(w) \sim \frac{\delta^k_j K^i{}_l(w) - \delta^i_l K^k{}_j(w)}{z-w} + \frac{2k}{(z-w)^2}\delta^i_l\delta^k_j \,, \tag{4.6}$$

where the normalization is chosen such that $k = -1/2$. The corresponding commutation rules read

$$[J_{ij,m}, \bar{J}^{kl}_n] \,=\, \delta^l_j K^k{}_{i,m+n} + 3 \text{ terms} - 4km\delta^k_{(i}\delta^l_{j)}\delta_{m+n,0} \,, \tag{4.7}$$

$$[K^i{}_{j,m}, \bar{J}^{kl}_n] \,=\, \delta^k_j\bar{J}^{il}_{m+n} + \delta^l_j\bar{J}^{ik}_{m+n} \,, \tag{4.8}$$

$$[K^i{}_{j,m}, J_{kl,n}] \,=\, -\delta^i_k J_{jl,m+n} - \delta^i_l J_{jk,m+n} \,, \tag{4.9}$$

$$[K^i{}_{j,m}, K^k{}_{l,n}] \,=\, \delta^k_j K^i{}_{l,m+n} - \delta^i_l K^k{}_{j,m+n} + 2km\delta^i_l\delta^k_j\delta_{m+n,0} \,. \tag{4.10}$$

[16] Doublet indices are raised and lowered with ϵ^{ij} and ϵ_{ij} defined so that $\epsilon_{ik}\epsilon^{jk} = \delta^j_i$. The symmetric Pauli matrices $(\sigma_r)_{ij} = (\sigma_r)_i{}^k\epsilon_{kj}$ and $(\sigma_r)^{ij} = \epsilon^{ik}(\sigma_r)_k{}^j$ obey the reality condition $((\sigma_r)_{ij})^* = -(\sigma_r)^{ij}$.

The $\widehat{\mathfrak{sp}}(4)_{-1/2}$ Sugawara stress tensor equals the canonical stress tensor of the symplectic bosons,

$$T = \frac{1}{2}\left((a_i\partial\bar{a}^i) - (\bar{a}^i\partial a_i)\right) \tag{4.11}$$

with $c = -2$. The relation between $(\bar{J}^{ij}, K^i{}_j, J_{ij})$ and the $O(3)$-covariant compact basis $(L_r^{\pm}, E, M_{rs})$ is given by

$$\bar{J}^{ij} = (\sigma^r)^{ij}L_r^+\,, \qquad J_{ij} = -(\sigma^r)_{ij}L_r^-\,, \tag{4.12}$$

$$K^i{}_j = \delta^i_j E + \tfrac{i}{2}(\sigma^{rs})_j{}^i M_{rs}\,. \tag{4.13}$$

The currents $K^i{}_j$ generate the $\widehat{\mathfrak{u}}(2)_{-1/2} \simeq \widehat{\mathfrak{su}}(2)_{-1/2} \oplus \widehat{\mathfrak{u}}(1)_{-1/2} \simeq \widehat{\mathfrak{so}}(3)_{-1/2} \oplus \widehat{\mathfrak{so}}(2)_{-1/2}$ subalgebra. There are also two independent $\widehat{\mathfrak{sp}}(2)_{-1/2} \simeq \widehat{\mathfrak{so}}(1,2)_{-1}$ subalgebras generated by the currents $(r' = +, 3, -)$

$$J^{r'(i)} = (J^{+(i)}, J^{3(i)}, J^{-(i)}) = \tfrac{1}{2}(\bar{J}^{ii}, K^i{}_i, J_{ii})\,, \tag{4.14}$$

for fixed $i = 1, 2$. The two $\mathfrak{sp}(2)$ spins are given in terms of the space-time energy and spin by

$$K^1{}_1 = E + M_3\,, \qquad K^2{}_2 = E - M_3\,, \tag{4.15}$$

where we use the following canonical basis for SO(3): $M_r = -\epsilon_{rst}M_{st}/2$, $[M_r, M_s] = i\epsilon_{rst}M_t$. To represent the symplectic bosons in Fock spaces, one expands them in modes

$$a_i(z) = \sum_{n\in\mathbb{Z}+\mu} z^{-n-1/2}a_{i,n}\,, \qquad \bar{a}^i(z) = \sum_{n\in\mathbb{Z}+\mu} z^{-n-1/2}\bar{a}^i_n\,, \tag{4.16}$$

where by definition $\mu \in \{0, 1/2\}$. The operator product (4.1) is equivalent to the oscillator algebra

$$[a_{m,i}, \bar{a}^j_n] = \delta^j_i\delta_{m+n,0}\,. \tag{4.17}$$

One next introduces *P-twisted oscillator vacua* $|[P]\rangle$ obeying

$$a_{i,n}|[P]\rangle = 0\,, \quad \text{for} \quad n \geq -\tfrac{P-1}{2}\,, \qquad \bar{a}^i_n|[P]\rangle = 0\,, \quad \text{for} \quad n \geq \tfrac{P+1}{2}\,, \tag{4.18}$$

where $P \in 2\mathbb{Z} + 2\mu$, giving rise to the P-twisted sectors

$$\widehat{\mathcal{F}}_{[P]} = \left\{ \prod_{n\geq\frac{P+1}{2}} (a_{i,-n})^{k_n} \prod_{n\geq-\frac{P-1}{2}} \left(\bar{a}^i_{-n}\right)^{\bar{k}_n} |[P]\rangle \right\}\,, \tag{4.19}$$

where $k_n, \bar{k}_n \in \mathbb{Z}_{\geq 0}$. The P-twisted normal order is defined by

$${}^{\times}_{\times}\mathcal{O}a_{i,n}{}^{\times}_{\times} = \begin{cases} \mathcal{O}a_{i,n} \text{ for } n \geq -\frac{P-1}{2} \\ a_{i,n}\mathcal{O} \text{ else} \end{cases}\,, \tag{4.20}$$

$$ {}^{\times}_{\times}\mathcal{O}\bar{a}^i_n{}^{\times}_{\times} = \begin{cases} \mathcal{O}\bar{a}^i_n \text{ for } n \geq \frac{P+1}{2} \\ \bar{a}^i_n\mathcal{O} \text{ else} \end{cases} , \tag{4.21} $$

where we note that the odd-twisted normal orderings send $a_{i,0}$ and $\bar{a}^i_0$ to the left and the right, respectively. The P-twisted Green's functions are given by

$$ a_i(z)\bar{a}^j(w) - {}^{\times}_{\times}a_i(z)\bar{a}^j(w){}^{\times}_{\times} = \frac{\left(\frac{z}{w}\right)^{P/2}\delta^j_i}{z-w} , \tag{4.22} $$

and the normal ordering only affects affects the energy operator $E = (\bar{a}^i a_i)/2$ and the stress energy tensor $T = (\partial\bar{a}^i a_i - \bar{a}^i\partial a_i)/2$. The 0-twisted normal order coincides with $(\cdot)$, and one can identify the 0-twisted sector with the standard NS sector, i.e.

$$ |[0]\rangle = |0\rangle , \tag{4.23} $$

is the $\widehat{\mathfrak{sp}}(4)_{-1/2}$ invariant state obeying $M_{AB,n}|0\rangle = L_n|0\rangle = 0$ for $n \geq 0$. The (± 1)-twisted sectors are analogs of the R-sector in the realization of $\widehat{\mathfrak{so}}(4)_1$ in terms of 4 real fermions with $c_{\text{fermion}} = 2$ (here the currents are based on anti-symmetric SO(4)-invariant bilinear forms; the central extension in $\widehat{\mathfrak{so}}(4)_1$ is fixed by equating the Sugawara and canonical stress tensors). In particular, the bosonic Fock space generated by the zero-modes $(a_{i,0}, \bar{a}^i_0)$ are analogs of the finite-dimensional spinor representations of the Clifford algebra with $h_{\text{spinor}} = 1/4$. Switching back to 4 real bosons reverses the signs in c and h and induces a non-trivial mixing between the stress tensor and the AdS energy (whose fermionic analog vanishes identically due to the odd statistics) with the result that

$$ E(z) = \tfrac{1}{2}{}^{\times}_{\times}\bar{a}^i a_i{}^{\times}_{\times} + \tfrac{P}{2}z^{-1} , \tag{4.24} $$

$$ T(z) = \tfrac{1}{2}{}^{\times}_{\times}(\partial\bar{a}^i a_i - \bar{a}^i\partial a_i){}^{\times}_{\times} - \tfrac{P}{2}{}^{\times}_{\times}\bar{a}^i a_i{}^{\times}_{\times}z^{-1} - \tfrac{P^2}{4}z^{-2} , \tag{4.25} $$

where $P = \pm 1$. In fact, the above expression is valid for any P, in agreement with (3.51) and (3.52). One way of showing this is to note that for $P = P' \bmod 2$ one can use simple reorderings of oscillators to go between the P-twisted and P'-twisted normal ordered forms of operators corresponding to states in the NS sector, i.e. monomials in the symplectic bosons. The KM charges and Virasoro generators can consequently be expanded in the P-twisted sector as follows

$$ \bar{J}^{ij}_n = \sum_{m\in\mathbb{Z}+\mu} {}^{\times}_{\times}\bar{a}^i_m\bar{a}^j_{n-m}{}^{\times}_{\times} , $$

$$ J_{ij,n} = \sum_{m\in\mathbb{Z}+\mu} {}^{\times}_{\times}a_{i,m}a_{j,n-m}{}^{\times}_{\times} , \tag{4.26} $$

$$ K^i{}_{j,n} = \sum_{m\in\mathbb{Z}+\mu} {}^{\times}_{\times}\bar{a}^i_m a_{j,n-m}{}^{\times}_{\times} + \tfrac{P}{2}\delta_{n,0}\delta^i_j , \tag{4.27} $$

$$ L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}+\mu} {}^{\times}_{\times}\left(m(\bar{a}^i_{n-m}a_{i,m} - \bar{a}^i_m a_{i,n-m}) - P\bar{a}^i_m a_{i,n-m}\right){}^{\times}_{\times} - \tfrac{P^2}{4}\delta_{n,0} . \tag{4.28} $$

The star conjugation $*$ and BPZ conjugation τ defined in (3.8) take the following form in the compact basis

$$\begin{aligned}
&(J_{ij,n})^* = -\bar{J}^{ij}_n\,, && (\bar{J}^{ij}_n)^* = -J_{ij,n}\,, && (K^i{}_{j,n})^* = -K^j{}_{i,n}\,,\\
&(J_{ij,n})^\tau = -J_{ij,-n}\,, && (\bar{J}^{ij}_n)^\tau = -\bar{J}^{ij}_{-n}\,, && (K^i{}_{j,n})^\tau = -K^i{}_{j,-n}\,,\\
&(J_{ij,n})^\dagger = \bar{J}^{ij}_{-n}\,, && (\bar{J}^{ij}_n)^\dagger = J_{ij,-n}\,, && (K^i{}_{j,n})^\dagger = K^j{}_{i,-n}\,.
\end{aligned} \tag{4.29}$$

The conjugations can be implemented at the level of the oscillator algebra by taking

$$(a_{i,n})^* = i\bar{a}^i_n\,, \qquad (\bar{a}^i_n)^* = ia_{i,n}\,, \tag{4.30}$$

$$(a_{i,n})^\tau = -ia_{i,-n}\,, \qquad (\bar{a}^i_n)^\tau = -i\bar{a}^i_{-n}\,, \tag{4.31}$$

$$(a_{i,n})^\dagger = \bar{a}^i_{-n}\,, \qquad (\bar{a}^i_n)^\dagger = a_{i,-n}\,. \tag{4.32}$$

The π map, defined by (3.9), takes the following form in the U(2)-covariant basis,

$$\pi(J_{ij,n}) = \bar{J}_{ij,n}\,, \qquad \pi(\bar{J}^{ij}_n) = J^{ij}_n\,, \qquad \pi(K^i{}_{j,n}) = K_j{}^i = \epsilon^{ik}K^l{}_k\epsilon_{lj}\,, \tag{4.33}$$

and can be implemented as

$$\pi(a_{i,n}) = \bar{a}_{i,n}\,, \qquad \pi(\bar{a}^i_n) = a^i_n\,. \tag{4.34}$$

Formally, the action of the conjugations and the π map on states can be defined as in (3.55) and (3.56).

4.2 Generating the spectrum by spectral flow

To describe the P-twisted sectors by means of the spectral flow operation, we define

$$|[P]\rangle = \Omega_P\left[|0\rangle\right]\,. \tag{4.35}$$

From the fact that $\Omega_P[a_i(z)|0\rangle]$ and $\Omega_P[\bar{a}^i(z)|0\rangle]$ are regular and non-vanishing at $z=0$ it follows that

$$(\Omega_P a_i \Omega_P^{-1})(z) = z^{-P/2}a_i(z)\,, \qquad (\Omega_P \bar{a}^i \Omega_P^{-1})(z) = z^{P/2}\bar{a}^i(z)\,, \tag{4.36}$$

or, in terms of modes,

$$\Omega_P a_{i,n}\Omega_P^{-1} = a_{i,n-P/2}\,, \qquad \Omega_P \bar{a}^i_n \Omega_P^{-1} = \bar{a}^i_{n+P/2}\,. \tag{4.37}$$

The spectral flows of $E(z)$ and $T(z)$ can then be calculated either by implementing the normal order using an auxiliary integration and following steps similar to those that led from (3.48) to (3.49), or by acting directly on the mode expansions in (4.27) and (4.28). The result can be written as

$$\Omega_P E(z)\Omega_P^{-1} = E(z) - \tfrac{P}{2}z^{-1}\,, \tag{4.38}$$

$$\Omega_P T(z)\Omega_P^{-1} = T(z) + \tfrac{P}{z}E(z) - \tfrac{P^2}{4}z^{-2}\,, \tag{4.39}$$

where the left-hand sides above are in P-twisted normal order, which ensures agreement with (4.24) and (4.25). It is now straightforward to verify that

$$\bar{J}^{ij}_n|[P]\rangle = 0\,, \qquad n \geq P\,, \tag{4.40}$$

$$J_{ij,n}|[P]\rangle = 0\,, \qquad n \geq -P\,, \tag{4.41}$$

$$\left(K^i{}_{j,n} - \tfrac{P}{2}\delta_{n,0}\delta^i_j\right)|[P]\rangle = 0\,, \qquad n \geq 0\,, \tag{4.42}$$

$$\left(L_n + \tfrac{P^2}{4}\delta_{n,0}\right)|[P]\rangle = 0\,, \qquad n \geq 0\,, \tag{4.43}$$

i.e. $|[P]\rangle$ corresponds to the $\mathfrak{so}(2,3)_{[P]}$-singlet P-twisted primary defined in (3.21) and (3.22). In addition to the singlet, there is a twisted primary spinor, given for $P \neq 0$ by

$$|[P]; -\tfrac{(|P|+1)^2}{4} + \tfrac{3}{4}; \tfrac{(P+\sigma)}{2}, \tfrac{1}{2})\rangle = \begin{cases} \bar{a}^i_{(P-1)/2}|[P]\rangle & P > 0 \\ a^i_{-(P+1)/2}|[P]\rangle & P < 0 \end{cases}\,, \tag{4.44}$$

and for $P = 0$ by the quartet (denoted here as a lowest weight state)

$$|[0]; \tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}\rangle = \left\{\bar{a}^i{}_{-\frac{1}{2}}|0\rangle\,,\ a^i{}_{-\frac{1}{2}}|0\rangle\right\}\,. \tag{4.45}$$

As shown in appendix A, there are no other primary states, so that

$$\begin{aligned}\widehat{\mathcal{F}}_{[P]} = {} & \widehat{\mathfrak{D}}_{[0];0}(0,0) \oplus \widehat{\mathfrak{D}}_{[0];\frac{1}{2}}(-\tfrac{1}{2}, \tfrac{1}{2}) \\ & \bigoplus_{P\neq 0}\left[\widehat{\mathfrak{D}}_{[P];-\frac{P^2}{4}}(\tfrac{P}{2}, 0) \oplus \widehat{\mathfrak{D}}_{[P];-\frac{(|P|+1)^2}{4}+\frac{3}{4}}(\tfrac{(P+\sigma)}{2}, \tfrac{1}{2})\right]\,.\end{aligned} \tag{4.46}$$

4.3 Oscillator realization of the multipleton subsectors

Let us make contact with Dirac's original oscillator realization of the $\mathfrak{sp}(4)$ singletons. For $P \neq 0$ we split the oscillators $(\bar{a}^i_n, a_{i,n})$ $(n \in \mathbb{Z} + P/2)$ into P-twisted non-zero modes with $|n| \geq (|P|+1)/2$ and zero modes with $|n| \leq (|P|-1)/2$. The zero modes can be assembled into $|P|$ sets of U(2)-covariant oscillators $(b_i(\xi), \bar{b}^i(\xi))$, $\xi = 1, \ldots, |P|$, obeying

$$[b_i(\xi), \bar{b}^j(\eta)] = \delta_{\xi\eta}\delta^j_i\,, \tag{4.47}$$

$$P > 0\ :\ b_i(\xi)|[P]\rangle = 0\,, \tag{4.48}$$

$$P < 0\ :\ \bar{b}^i(\xi)|[P]\rangle = 0\,. \tag{4.49}$$

This can be described geometrically in the complex plane as the conformal transformation $z = \exp(\zeta/P)$, which maps the symplectic bosons to

$$b_i(\zeta) \equiv \left(\frac{dz}{d\zeta}\right)^{1/2} a_i(z(\zeta)) = \frac{1}{\sqrt{P}}\sum_n e^{-\frac{n\zeta}{P}} a_{i,n}\,, \tag{4.50}$$

$$\bar{b}^i(\zeta) \equiv \left(\frac{dz}{d\zeta}\right)^{1/2} \bar{a}^i(z(\zeta)) = \frac{1}{\sqrt{P}}\sum_n e^{-\frac{n\zeta}{P}} \bar{a}^i_n\,, \tag{4.51}$$

followed by a truncation to zero modes, defined by

$$b_i^{(P)}(\zeta) \;=\frac{1}{\sqrt{P}}\sum_{|n|\leq(|P|-1)/2} e^{-\frac{n\zeta}{P}}a_{i,n}\,,\qquad \bar{b}^{(P)i}(\zeta) \;=\frac{1}{\sqrt{P}}\sum_{|n|\leq(|P|-1)/2} e^{-\frac{\zeta}{P}}\bar{a}^i_n\,, \quad (4.52)$$

after which one identifies

$$b_i(\xi)=b_i^{(P)}(2\pi i\xi)\,,\qquad \bar{b}^i(\xi) \;=\; \bar{b}^{(P)i}(2\pi i\xi)\;. \quad (4.53)$$

The zero modes yields a subspace of $\mathcal{F}_{[P]}\subset\widehat{\mathcal{F}}_{[P]}$ given by

$$\begin{aligned}\mathcal{F}_{[P]} &= \begin{cases}\left\{\prod_{\xi,i}\bar{b}^i(\xi)|[P]\rangle\right\}\text{ for }P>0\,,\\ \left\{\prod_{\xi,i}b_i(\xi)|[P]\rangle\right\}\text{ for }P<0\end{cases}\\ &\simeq\left(\mathcal{F}^{\mathrm{sign}(P)}\right)^{\otimes|P|}\,,\qquad P\neq0\;,\end{aligned} \quad (4.54)$$

where $\mathcal{F}^+$ denotes the standard Fock space and $\mathcal{F}^-$ the anti-Fock space of a $\mathrm{U}(2)$-covariant oscillator $(b_i,\bar{b}^i)$ obeying $[b_i,\bar{b}^j]=\delta_i^j$. The latter is obtained by acting with b_i on an anti-vacuum $|0\rangle^-$ obeying $\bar{b}^i|0\rangle^-=0$. The algebra $\mathfrak{sp}(4)$ is represented in $\mathcal{F}^\pm$ by

$$\bar{J}^{(b)ij}=\bar{b}^i\bar{b}^j\,,\qquad K^{(b)i}{}_j \;=\; \bar{b}^ib_j+\tfrac{1}{2}\delta^i_j \;=\; b_j\bar{b}^i-\tfrac{1}{2}\delta^i_j\;,\qquad J^{(b)}_{ij} \;=\; b_ib_j\,, \quad (4.55)$$

leading to the decompositions

$$\mathcal{F}^\pm=\mathfrak{D}^\pm_0\oplus\mathfrak{D}^\pm_{1/2}\;, \quad (4.56)$$

where the scalar and spinor singletons and anti-singletons are realized as $(m=0,1/2)$

$$\mathfrak{D}^+_m=\left\{\bar{b}^{i_1}\cdots\bar{b}^{i_{2(n+m)}}|[1]\rangle\right\}^\infty_{n=0}\,,\qquad \mathfrak{D}^-_m=\left\{b_{i_1}\cdots b_{i_{2(n+m)}}|[-1]\rangle\right\}^\infty_{n=0}\;. \quad (4.57)$$

The ground states $|1/2,0\rangle^\pm$ and $|1,1/2\rangle^+_i$ and $|1,1/2\rangle^-_i$ obey

$$\text{Scalar:}\quad K^{(b)i}{}_j|\tfrac{1}{2},0\rangle^\pm \;=\; \pm\tfrac{1}{2}\delta^i_j|\tfrac{1}{2},0\rangle^\pm\,, \quad (4.58)$$

$$J^{(b)}_{ij}|\tfrac{1}{2},0\rangle^+ \;=\; 0\,,\qquad \bar{J}^{(b)ij}|\tfrac{1}{2},0\rangle^- \;=\; 0\,, \quad (4.59)$$

$$\text{Spinor:}\quad K^{(b)i}{}_j|1,\tfrac{1}{2}\rangle^+_k \;=\; \pm\left(\tfrac{1}{2}\delta^i_j|1,\tfrac{1}{2}\rangle^+_k+\epsilon_{jk}|1,\tfrac{1}{2}\rangle^{+i}\right)\;, \quad (4.60)$$

$$J^{(b)}_{ij}|1,\tfrac{1}{2}\rangle^+_k \;=\; 0\;,\qquad \bar{J}^{(b)ij}|1,\tfrac{1}{2}\rangle^-_k \;=\; 0\;. \quad (4.61)$$

Thus, for $P\neq0$ the space $\mathcal{F}_{[P]}$ consists of all possible P-tupletons built from scalar as well as spinor (anti-)singletons of $\mathfrak{sp}(4)$ (we note that the permutation group $\mathcal{S}_P$, which acts on ξ, and the Virasoro generator L_0 do not commute). For $P=0$ we instead define

$$\mathcal{F}_{[0]} \;=\; \mathfrak{D}_{[0];0}(0,0)\oplus\mathfrak{D}_{[0];\frac{1}{2}}(-\tfrac{1}{2},\tfrac{1}{2})\,, \quad (4.62)$$

which form a finite-dimensional reducible $\mathfrak{so}(2,3)$ representation space.

Let us examine the cases $P = \pm 1$ and $P = \pm 2$ in more detail. In $\widehat{\mathcal{F}}_{[\pm 1]}$ we find the subspace $\mathcal{F}_{[\pm 1]}$ generated by the zero modes $(b_i(1), \bar{b}^i(1)) = (a_{i,0}, \bar{a}^i_0)$. Here, the KM charges $(\bar{J}^{ij}_n, K^i{}_{j,n}, J_{ij,n})$ vanish for $n > 0$ and reduce to the oscillator representation (4.55) for $n = 0$, leading to an oscillator realization of the $\mathfrak{sp}(4)$ singletons and anti-singletons of the form given in (4.57). For $P = \pm 2$, the zero modes

$$b_i(1) = \frac{i}{\sqrt{2}}(a_{i,-1/2} - a_{i,1/2})\,, \qquad \bar{b}^i(1) \;=\; \frac{i}{\sqrt{2}}(\bar{a}^i_{-1/2} - \bar{a}^i_{1/2})\,, \tag{4.63}$$

$$b_i(2) = \frac{1}{\sqrt{2}}(a_{i,-1/2} + a_{i,1/2})\,, \qquad \bar{b}^i(2) \;=\; \frac{1}{\sqrt{2}}(\bar{a}^i_{-1/2} + \bar{a}^i_{1/2})\,, \tag{4.64}$$

generate the subspaces $\mathcal{F}_{[\pm 2]} \subset \widehat{\mathcal{F}}_{[\pm 2]}$, that decompose into massless $\mathfrak{sp}(4)$ representations in accordance with (2.24). Diagonalizing L_0, one finds that

$$\mathcal{F}_{[\pm 2]} \simeq (\mathfrak{D}^{\pm}_0 \oplus \mathfrak{D}^{\pm}_{1/2})^{\otimes 2}\,, \tag{4.65}$$

in agreement with (4.54) and (4.56). The ground states are given by

$$|[2]; \pm m - 1; m + 1, m\rangle^{i(2m)} = \bar{a}^{i_1}_{\mp\frac{1}{2}} \cdots \bar{a}^{i_{2m}}_{\mp\frac{1}{2}} |[2]\rangle\,, \tag{4.66}$$

$$|[-2]; \pm m - 1; -m - 1, m\rangle_{i(2m)} = a_{i_1,\mp\frac{1}{2}} \cdots a_{i_{2m},\mp\frac{1}{2}} |[-2]\rangle\,, \tag{4.67}$$

$$|[2]; -1; 2, 0\rangle = \epsilon_{ij} \bar{a}^i_{\frac{1}{2}} \bar{a}^j{}_{-\frac{1}{2}} |[2]\rangle\,, \tag{4.68}$$

$$|[-2]; -1; -2, 0\rangle = \epsilon^{ij} a_{i,\frac{1}{2}} a_{j,-\frac{1}{2}} |[-2]\rangle\ . \tag{4.69}$$

We note that all ground states are built from oscillators with the same Virasoro mode numbers except the massless pseudo-scalar ground state, which we can also write as $|[2]; -1; 2, 0\rangle = \epsilon_{ij} \bar{J}^{ki}_1 K^j{}_{k,-1} |[2]; -1; 1, 0\rangle$ in agreement with (3.61). The KM charges $(\bar{J}^{ij}_n, K^i{}_{j,n}, J_{ij,n})$ with $n > 1$ vanish in $\mathcal{F}_{[\pm 2]}$. The ground states in $\mathcal{F}_{[2];-}$ can be written as strings of $\bar{J}^{ij}_1$ charges acting on the scalar ground state $|[2]\rangle$ and the spinor ground state $\bar{a}^i_{-1/2}|[2]\rangle$. On the other hand, $\bar{J}^{ij}_{-1}$ yields the massless higher-spin ground states in $\mathcal{F}_{[2];+}$ together with extra contributions involving oscillator non-zero modes that can be cancelled by adding terms involving $K^i{}_{j,-1}$ of the form given in (3.34).

So far, the realization of the $\widehat{\mathfrak{so}}(2,3)_{-1/2}$ in terms of symplectic bosons supports the claim made in (3.75), with the identification

$$\mathfrak{D}_{[P]} = \mathcal{F}_{[P]}\ . \tag{4.70}$$

Next, we wish to continue and analyze in more detail the fusion rules and locality properties of the operator products using a free-field realization.

5. Free-field realization of the extended $\widehat{\mathfrak{sp}}(4)$ model

In this section we use a free-field realization to analyze the fusion rules and locality properties of the operator algebra of the extended $\widehat{\mathfrak{sp}}(4)$ model introduced in section 4. This realization also provides a very simple implementation of the spectral flow Ω_P.

5.1 Partial bosonization

To construct the $\widehat{\mathfrak{sp}}(4)_{-1/2}$ model out of free fields one in principle needs to introduce four real free bosons, (φ^i, σ^i), $i = 1, 2$, in a space with signature $(--++)$ and background charges $(0, 0, i/\sqrt{2}, i/\sqrt{2})$. However, when it comes to describing the states of the Fock spaces introduced in the previous section, it is more practical to work in a partially bosonized picture analogous to that of the $\widehat{\mathfrak{sp}}(2)_{-1/2}$ model worked out in [21] (see also [50]). Here, one retains the two time-like components φ^i $(i = 1, 2)$, defined by

$$\varphi^i(z)\varphi^j(w) \sim \delta^{ij} \ln(z-w)\,, \qquad T_\varphi \;=\; \tfrac{1}{2}(\partial\varphi^i\partial\varphi^i)\,, \tag{5.1}$$

and replaces the two space-like bosons by two sets of Grassmann odd weight $(0, 1)$ fields (ξ^i, η^i) $(i = 1, 2)$, defined by

$$\xi^i(z)\eta^j(w) \sim -\eta^j(w)\xi^i(z) \;\sim\; \frac{\delta^{ij}}{z-w}\,, \qquad T_{(\xi,\epsilon)} \;=\; (\partial\xi^i\eta^i)\,. \tag{5.2}$$

Giving these fields the following mode expansions

$$\varphi^i(z) \,=\, q^i - ip^i \ln z + i\sum_{n\neq 0}\frac{\alpha^i_n}{n} z^{-n}\,, \tag{5.3}$$

$$\xi^i(z) = \sum_{n\in\mathbb{Z}} \xi^i_n z^{-n}\,, \qquad \eta^i(z) \;=\; \sum_{n\in\mathbb{Z}} \eta^i_n z^{-n-1}\,, \tag{5.4}$$

the operator product expansions correspond to the commutation rules

$$[q^i, p^j] = -i\delta^{ij}\,, \qquad [\alpha^i_m, \alpha^j_n] \;=\; -m\delta^{ij}\delta_{m+n,0}\,, \qquad \{\xi^i_m, \eta^j_n\} \;=\; \delta^{ij}\delta_{m+n,0}\,, \tag{5.5}$$

and the NS vacuum is the state obeying

$$\alpha^i_n|0\rangle \;=\; \xi^i_{n+1}|0\rangle = \eta^i_n|0\rangle \;=\; 0 \qquad \text{for } n \geq 0\,, \tag{5.6}$$

where $\alpha^i_0 \equiv p^i$, so that $\varphi^i(z)|0\rangle$, $\xi^i(z)|0\rangle$ and $\eta^i(z)|0\rangle$ are regular and non-vanishing at $z = 0$. For operators that do not depend on q^i, i.e. $\mathcal{O}(\partial\varphi^i, \xi^i, \eta^i)$, the normal ordering $:\mathcal{O}:$ is defined via the standard prescription $(\cdot)$, which is an associative and commutative operation for free fields. The ordering of operators that also depend on q^i is defined by declaring that $:q^i\mathcal{O} := q^i:\mathcal{O}:$. The total Virasoro generators are given by

$$L_n = -\tfrac{1}{2}\sum_{m\in\mathbb{Z}} \alpha^i_{n-m}\alpha^i_m - \sum_{m\in\mathbb{Z}} m\xi^i_m\eta^i_{n-m}\,, \qquad n \neq 0\,, \tag{5.7}$$

$$L_0 = -\tfrac{1}{2}p^ip^i - \sum_{m>0}\alpha^i_{-m}\alpha^i_m + \sum_{m>0} m\left(\xi^i_{-m}\eta^i_m + \eta^i_{-m}\xi^i_m\right)\,, \tag{5.8}$$

and the total central charge $c = 2\times 1 + 2\times(-2) = -2$. The vertex operators

$$V_\lambda \,=\, :c_\lambda(p)\exp(-i\lambda^i\varphi^i):\,, \qquad c_\lambda(p) \;=\; \exp\left(\frac{i\pi}{2}\epsilon^{ij}\lambda^ip^j\right) \tag{5.9}$$

have conformal weights $h_\lambda = -\frac{1}{2}\lambda^2$, where $\lambda^2 = \lambda^i\lambda^i$, and create states carrying finite momenta,

$$|\lambda\rangle \;=\; e^{-i\lambda^i q^i}|0\rangle\,, \qquad (p^i - \lambda^i)|\lambda\rangle \;=\; 0\;. \tag{5.10}$$

The vertex operators obey the composition rule

$$V_\lambda(z)V_{\lambda'}(w) = e^{\frac{i\pi}{2}\epsilon^{ij}\lambda^i\lambda'^j}(z-w)^{-\lambda\lambda'} : c_{\lambda+\lambda'}(p)V_\lambda(z)V_{\lambda'}(w) : \;. \tag{5.11}$$

Reversing the order of the product and continuing analytically using $w - z = e^{\pm i\pi}(z-w)$ yields the following monodromy matrices,

$$V_\lambda(z)V_{\lambda'}(w) = M^{\pm}_{\lambda,\lambda'}V_{\lambda'}(w)V_\lambda(z)\;, \tag{5.12}$$

$$M^{\pm}_{\lambda,\lambda'} = \exp\left(i\pi(\epsilon^{ij} \pm \delta^{ij})\lambda^i\lambda'^j\right)\;. \tag{5.13}$$

The realization of the symplectic bosons now takes the form

$$a_i = V_{\lambda_{(i)}}\eta^i \;=: c_{\lambda_{(i)}}(p)e^{-i\lambda^j_{(i)}\varphi^j} : \eta^i\,, \tag{5.14}$$

$$\bar a^i = V_{-\lambda_{(i)}}\partial\xi^i \;=: c_{-\lambda_{(i)}}(p)e^{i\lambda^j_{(i)}\varphi^j} : \partial\xi^i\,, \tag{5.15}$$

where $\lambda_{(1)} = (1,0)$ and $\lambda_{(2)} = (0,1)$. The expansions (4.16) correspond to momenta in the lattice

$$\lambda^i \in \mathbb{Z} + \mu\,, \qquad i = 1,2\;. \tag{5.16}$$

The $*$ map (4.30), which can be written as $(a_i(\bar z))^* = i\bar a^i(z)$ and $(\bar a^i(\bar z))^* = ia_i(z)$, requires

$$(\varphi^i(\bar z))^* = \varphi^i(z)\,, \qquad (\eta^i(\bar z))^* \;=\; i\partial\xi^i(z)\;. \tag{5.17}$$

The latter map can be implemented in the space of *reduced states* obeying

$$\eta^i_0|\Psi\rangle = 0\;. \tag{5.18}$$

Here, the $*$ map and conjugations take the form[17]

$$(q^i)^* = q^i\,, \qquad (\alpha^i_n)^* \;=\; -\alpha^i_n\,, \qquad (\eta^i_n)^* \;=\; -in\xi^i_n\,, \qquad (\xi^i_n)^* \;=\; -\frac{i}{n}\eta^i_n\,, \tag{5.19}$$

$$(q^i)^\tau = q^i\,, \qquad (\alpha^i_n)^\tau = -\alpha^i_{-n}\,, \qquad (\eta^i_n)^\tau \;=\; -\eta^i_{-n}\,, \qquad (\xi^i_n)^\tau \;=\; \xi^i_{-n} \tag{5.20}$$

$$(q^i)^\dagger = q^i\,, \qquad (\alpha^i_n)^\dagger \;=\; \alpha^i_{-n}\;, \qquad (\eta^i_n)^\dagger \;=\; in\xi^i_{-n}\,, \qquad (\xi^i_n)^\dagger \;=\; -\frac{i}{n}\eta^i_{-n}\;. \tag{5.21}$$

where $n \neq 0$ for the fermionic oscillators. In the realization (5.14), the $\widehat{\mathfrak{sp}}(4)_{-1/2}$ currents read (*cf.* (4.3)–(4.6))

$$J_{ij} = V_{\lambda_{(i)}+\lambda_{(j)}}\chi_{ij} \;=: c_{\lambda_{(i)}+\lambda_{(j)}}(p)e^{-i(\varphi^i+\varphi^j)} : \chi_{ij}\,, \tag{5.22}$$

$$\bar J^{ij} = V_{-\lambda_{(i)}-\lambda_{(j)}}\bar\chi^{ij} \;=: c_{-\lambda_{(i)}-\lambda_{(j)}}(p)e^{i(\varphi^i+\varphi^j)} : \bar\chi^{ij}\,, \tag{5.23}$$

$$K^i{}_j = V_{\lambda_{(i)}-\lambda_{(j)}}\chi^i{}_j \;=: c_{\lambda_{(i)}-\lambda_{(j)}}(p)e^{i(\varphi^j-\varphi^i)} : \chi^i{}_j\,, \tag{5.24}$$

[17] We define $(AB)^\tau = (-1)^{AB}B^\tau A^\tau$ and $(AB)^\dagger = (-1)^{AB}B^\dagger A^\dagger$.

where

$$\chi_{ij} = \begin{cases} \partial\eta^i\eta^j\,, & i=j \\ \eta^i\eta^j\,, & i\neq j \end{cases}\,, \qquad \bar{\chi}^{ij} = \begin{cases} \partial^2\xi^i\partial\xi^j\,, & i=j \\ \partial\xi^i\partial\xi^j\,, & i\neq j \end{cases}\,,$$

$$\chi^i{}_j = \begin{cases} -i\partial\varphi^i\,, & i=j \\ \partial\xi^2\eta^1\,, & i=2,\ j=1 \\ \partial\xi^1\eta^2\,, & i=1,\ j=2 \end{cases}\,. \tag{5.25}$$

We note that while the cocycle factors $c_\lambda(p)$ play a non-trivial role in reproducing the correct monodromy properties of the symplectic bosons (5.14) and (5.15), they do not contribute phase factors to the operator product between the currents. Next, we shall turn to the bosonization of the twist fields.

5.2 Singleton twist fields and their fusion rules

The singleton and anti-singleton states in $\mathcal{F}_{[\pm1]}$ correspond to a set of twist fields

$$\Sigma^{(e,m)}_{[\pm1]} = \left\{\Sigma^{(e,m;\ell)}_{[\pm1];-1/4}\right\}^m_{\ell=-m}\,, \qquad e \;=\; \pm(m+\tfrac{1}{2})\,, \tag{5.26}$$

where $m\in\mathbb{Z}_{\geq0}$ and $m\in\mathbb{Z}_{\geq0}+1/2$ for scalar and spinor singletons, respectively, that generate anti-periodic branch cuts in the symplectic bosons in accordance with (4.18) for $P=\pm1$. This requirement determines the twist fields, and one finds (*cf.* [21])

$$\Sigma^{(\pm m\pm1/2,m;\ell)}_{[\pm1];-1/4} = V_{\mp(m+\ell+\frac{1}{2},m-\ell+\frac{1}{2})}\sigma^{(m;\ell)}_{\pm}\,, \tag{5.27}$$

where

$$\sigma^{(m;\ell)}_{+} = (\partial\xi^1\partial^2\xi^1\cdots\partial^{m+\ell}\xi^1)(\partial\xi^2\partial^2\xi^2\cdots\partial^{m-\ell}\xi^2)\,, \tag{5.28}$$

$$\sigma^{(m;\ell)}_{-} = (\eta^1\partial\eta^1\cdots\partial^{m+\ell-1}\eta^1)(\eta^2\partial\eta^2\cdots\partial^{m-\ell-1}\eta^2)\,, \tag{5.29}$$

with the convention that $\partial\xi^1\cdots\partial^0\xi^1\equiv1$ idem ξ^2, η^1 and η^2. Indeed, the conformal weights $h\left[\Sigma^{(e,m)}_{[\pm1]}\right]=-\frac{1}{4}$, since $h\left[\sigma^{(m;\ell)}_{\pm}\right]=m(m+1)+\ell^2$ and $h\left[e^{i(m\pm\ell+1/2)\varphi^i}\right]=-\frac{1}{2}(m\pm\ell+\frac{1}{2})^2$. In particular, the products $J_{ij}(z)\Sigma^{(\pm1/2,0)}_{[1];-1/4}(w)$, $\bar{J}^{ij}(z)\Sigma^{(\pm1/2,0)}_{[-1];-1/4}(w)$ and $M_3(z)\Sigma^{(\pm1/2,0)}_{[1];-1/4}(w)$, where $M_3=(K^1{}_1-K^2{}_2)/2$, have trivial expansions, while

$$E(z)\Sigma^{(\pm1/2,0)}_{[\pm1];-1/4}(w) \sim \frac{\pm1/2}{z-w}\Sigma^{(\pm1/2,0)}_{[\pm1];-1/4}(w)\,, \tag{5.30}$$

where $E=(K^1{}_1+K^2{}_2)/2$. Similarly, $\Sigma^{(\pm1,1/2;\ell)}_{[\pm1];-1/4}$ ($\ell=\pm1/2$) correspond to the spinor ground states. We note that the anti-periodicity of the branch cut, i.e.

$$a_i(e^{2\pi i}z)\Sigma^{(e,m)}_{[\pm1]}(w) \;=\; -a_i(z)\Sigma^{(e,m)}_{[\pm1]}(w)\,, \tag{5.31}$$

$$a^i(e^{2\pi i}z)\Sigma^{(e,m)}_{[\pm1]}(w) \;=\; -a^i(z)\Sigma^{(e,m)}_{[\pm1]}(w)\,, \tag{5.32}$$

for $|z|>|w|$, does not rely on the cocycle factor.

Turning to the fusion rules in the $|P \leq 1$ sector, some of the simplest cases are

$$\Sigma^{(1/2,0)}_{[1];-1/4}(z)\Sigma^{(-1/2,0)}_{[-1];-1/4}(w) \sim (z-w)^{1/2}\,,$$
$$\Sigma^{(1/2,0)}_{[1];-1/4}(z)\Sigma^{(-3/2,1;1)}_{[-1];-1/4}(w) \sim (z-w)^{3/2}J_{11}(w)\,, \tag{5.33}$$

where the cocycle factors have been suppressed. More generally, for $m+\ell \geq m'+\ell'$ and $m-\ell \geq m'-\ell'$ one finds that

$$\Sigma^{(e,m;\ell)}_{[1];-1/4}(z)\Sigma^{(e,m';\ell')}_{[-1];-1/4}(w) \propto (z-w)^{m-m'+1/2} : (\bar{a}^1)^{m+\ell-m'-\ell'}(\bar{a}^2)^{m-\ell-m'+\ell'} : (w) \tag{5.34}$$

where the right-hand sides are descendants in the NS sector that can be written in terms of undifferentiated currents. Related expansions hold for other relations among (m,ℓ,m',ℓ'). Similarly, the products of twist fields from $\widehat{\mathfrak{D}}_{[1];-1/4}(1/2,0)$ and and $\widehat{\mathfrak{D}}_{[0];1/2}(-1/2,1/2)$ belong to $\widehat{\mathfrak{D}}_{[1];-1/4}(1,1/2)$; for example

$$\bar{a}^1(z)\Sigma^{(1/2,0)}_{[1];-1/4}(w) \sim \frac{\Sigma^{(1,1/2;1/2)}_{[1];-1/4}(w)}{(z-w)^{1/2}}\,, \qquad \bar{a}^2(z)\Sigma^{(1/2,0)}_{[1];-1/4}(w) \sim \frac{\Sigma^{(1,1/2;-1/2)}_{[1];-1/4}(w)}{(z-w)^{1/2}}\,. \tag{5.35}$$

Proceeding in this fashion, it is straightforward to verify the following fusion rules within the $|P| \leq 1$ sector:

$$\widehat{\mathfrak{D}}_{[\pm1]}(\pm\tfrac{1}{2},0) \times \widehat{\mathfrak{D}}_{[\mp1]}(\mp\tfrac{1}{2},0) = \widehat{\mathfrak{D}}_{[0]}(0,0)\,, \tag{5.36}$$
$$\widehat{\mathfrak{D}}_{[\pm1]}(\pm1,\tfrac{1}{2}) \times \widehat{\mathfrak{D}}_{[\mp1]}(\mp1,\tfrac{1}{2}) = \widehat{\mathfrak{D}}_{[0]}(0,0)\,, \tag{5.37}$$
$$\widehat{\mathfrak{D}}_{[\pm1]}(\pm\tfrac{1}{2},0) \times \widehat{\mathfrak{D}}_{[\mp1]}(\mp1,\tfrac{1}{2}) = \widehat{\mathfrak{D}}_{[0]}(-\tfrac{1}{2},\tfrac{1}{2})\,, \tag{5.38}$$
$$\widehat{\mathfrak{D}}_{[\pm1]}(\pm\tfrac{1}{2},0) \times \widehat{\mathfrak{D}}_{[0]}(-\tfrac{1}{2},\tfrac{1}{2}) = \widehat{\mathfrak{D}}_{[\pm1]}(\pm1,\tfrac{1}{2})\,, \tag{5.39}$$
$$\widehat{\mathfrak{D}}_{[\pm1]}(\pm1,\tfrac{1}{2}) \times \widehat{\mathfrak{D}}_{[0]}(-\tfrac{1}{2},\tfrac{1}{2}) = \widehat{\mathfrak{D}}_{[\pm1]}(\pm\tfrac{1}{2},0)\,, \tag{5.40}$$
$$\widehat{\mathfrak{D}}_{[0]}(-\tfrac{1}{2},\tfrac{1}{2}) \times \widehat{\mathfrak{D}}_{[0]}(-\tfrac{1}{2},\tfrac{1}{2}) = \widehat{\mathfrak{D}}_{[0]}(0,0)\,, \tag{5.41}$$

where the conformal dimensions are suppressed. Next, let us examine how the product of two singleton twist fields closes on a massless twist field.

5.3 Massless sector

The formal analysis in section 3 implies that the product of two singleton twist fields in the $P=1$ sector closes on a twist field in the $P=2$ sector, forming an affine version of the Flato-Frønsdal formulae (2.22)-(2.24). As an example, let us consider the product of two singleton ground-state twist fields:

$$\Sigma^{(1/2,0)}_{[1];-1/4}(z)\Sigma^{(1/2,0)}_{[1];-1/4}(w) \sim \frac{1}{(z-w)^{1/2}}\Sigma^{(1,0)}_{[2];-1}(w)\,. \tag{5.42}$$

As the notation indicates, the field

$$\Sigma^{(1,0)}_{[2];-1} \equiv \Sigma^{(1,0;0)}_{[2];-1} = V_{-(1,1)} =: c_{-(1,1)}e^{i(\varphi^1+\varphi^2)} :\,, \tag{5.43}$$

is the ground state of the affine massless scalar representation $\widehat{\mathfrak{D}}_{[2];-1}(1,0)$, obeying

$$\begin{aligned} E(z)\Sigma^{(1,0)}_{[2];-1}(w) &\sim \frac{1}{z-w}\Sigma^{(1,0)}_{[2];-1}(w)\,, \\ M_3(z)\Sigma^{(1,0)}_{[2];-1}(w) &\sim 0\,, \end{aligned} \tag{5.44}$$

$$\begin{aligned} J_{ij,n}\Sigma^{(1,0)}_{[2];-1} &= 0 \quad \text{for } n \geq -2\,, \\ \bar{J}^{ij}_n\Sigma^{(1,0)}_{[2];-1} &= 0 \quad \text{for } n \geq 2\,, \end{aligned} \tag{5.45}$$

in accordance with (4.40) — (4.43). Another illustrative example is

$$\Sigma^{(1/2,0)}_{[1];-1/4}(z)\Sigma^{(3/2,1;0)}_{[1];-1/4}(w) \sim \frac{\Sigma^{(2,1;0)}_{[2];-2}(w)}{(z-w)^{3/2}} + \frac{\frac{1}{4}:c_{-(2,2)}\partial\xi^1\partial\xi^2\partial e^{2i(\varphi^1+\varphi^2)}:(w)}{(z-w)^{1/2}}\,, \tag{5.46}$$

where the leading term is given by

$$\Sigma^{(2,1;0)}_{[2];-2} = :c_{-(2,2)}\partial\xi^1\partial\xi^2 e^{2i(\varphi^1+\varphi^2)}:\ . \tag{5.47}$$

From the operator product expansions

$$\begin{aligned} E(z)\Sigma^{(2,1;0)}_{[2];-2}(w) &\sim \frac{2}{z-w}\Sigma^{(2,1;0)}_{[2];-2}(w) \\ , M_3(z)\Sigma^{(2,1;0)}_{[2];-2}(w) &\sim 0\ , \end{aligned} \tag{5.48}$$

$$J_{ij,0}\Sigma^{(2,1;0)}_{[2];-2} = 0\,, \tag{5.49}$$

it follows that $\Sigma^{(2,1;0)}_{[2];-2}$ can be identified with the $\ell = 0$ ground state of the massless vector representation $\mathfrak{D}_{[2];-2}(2,1)$ defined in (3.27). The subleading term in (5.46) is an admixture of $L_{-1}\Sigma^{(2,1;0)}_{[2];-2}$ and $\bar{J}^{12}_0\Sigma_{[2];-1}(1,0)$. More generally, the twist fields $\Sigma^{(\pm m\pm1,m;\ell)}_{[\pm2];-m-1}$ corresponding to the *massless spin-m ground states* of $\mathfrak{D}_{[\pm2];-1-m}(\pm m\pm1,m)$ with minimal conformal weight $h = -1-m$, given in (3.33), arise as the leading terms in the expansions of products of arbitrary singleton-valued twist fields (these are standard composite massless representations of the $\mathfrak{sp}(4)$ generated by $M_{AB,0}$; see discussion below (3.27)). The result is

$$\Sigma^{(\pm m\pm1,m;\ell)}_{[\pm2];-m-1} = V_{\mp(m+\ell+1,m-\ell+1)}\sigma^{(m;\ell)}_{\pm}\ , \tag{5.50}$$

where $\sigma^{(m;\ell)}_{+}$ and $\sigma^{(m;\ell)}_{-}$ are defined in (5.28) and (5.29), respectively. One can verify that (5.50) agrees with (3.33) and enjoys the properties of a (±2)-twisted primary field, *viz.*

$$E(z)\Sigma^{(\pm m\pm1,m;\ell)}_{[\pm2];-m-1}(w) \sim \frac{\pm(m+1)}{z-w}\Sigma^{(\pm m\pm1,m;\ell)}_{[\pm2];-m-1}(w)\,, \tag{5.51}$$

$$M_3(z)\Sigma^{(\pm m\pm1,m;\ell)}_{[\pm2];-m-1}(w) \sim \frac{\pm\ell}{z-w}\Sigma^{(\pm m\pm1,m;\ell)}_{[\pm2];-m-1}(w)\,, \tag{5.52}$$

$$\begin{aligned} J_{ij,0}\Sigma^{(m+1,m;\ell)}_{[2];-m-1}(z) &= 0\,, \\ \bar{J}^{ij}_0\Sigma^{(-m-1,m;\ell)}_{[-2];-m-1}(z) &= 0\ . \end{aligned} \tag{5.53}$$

The fusions between $P=2$ fields and $P=-2$ fields produce descendants either to the identity or the quartet representation. For example we have that

$$\Sigma^{(1,0)}_{[2];-1}(z)\Sigma^{(-1,0)}_{[-2];-1}(w) \;\sim\; (z-w)^2\,, \tag{5.54}$$

$$\Sigma^{(1,0)}_{[2];-1}(z)\Sigma^{(-3/2,1/2;1/2)}_{[-2];-3/2}(w) \;\sim\; (z-w)^3 a_1(w)\,, \tag{5.55}$$

$$\Sigma^{(2,1;0)}_{[2];-2}(z)\Sigma^{(-1,0)}_{[-2];-1}(w) \;\sim\; (z-w)^4 \bar{J}^{12}(w)\,, \tag{5.56}$$

where the symplectic boson $a_1(z)$ belongs to $\widehat{\mathfrak{D}}_{[0];1/2}(-1/2,1/2)$. It is also interesting to examine the fusion between twist fields with $P=2$ and $P=-1$. For instance,

$$\Sigma^{(1,0)}_{[2];-1}(z)\Sigma^{(-1/2,0)}_{[-1];-1/4}(w) \;\sim\; (z-w)\Sigma^{(1/2,0)}_{[1];-1/4}(w)\,, \tag{5.57}$$

$$\Sigma^{(1,0)}_{[2];-1}(z)\Sigma^{(-1,1/2;1/2)}_{[-1];-1/4}(w) \;\sim\; -2(z-w)^2(J_{11,-1}\Sigma^{(1,1/2;1/2)}_{[1];-1/4})(w)\,, \tag{5.58}$$

$$\Sigma^{(2,1;0)}_{[2];-2}(z)\Sigma^{(-5/2,3/2;3/2)}_{[-1];-1/4}(w) \;\sim\; 6(z-w)^4(J_{11,-2}\Sigma^{(1,1/2;-1/2)}_{[1];-1/4})(w)\ . \tag{5.59}$$

In summary, the free-field realization confirms the fusion rules (3.59) derived in section 3 using spectral flow.

5.4 On massive sector and realization of spectral flow

In order to demonstrate how the spectral flow operation acts in the free field basis, let us begin by looking more closely at the twist fields $\Sigma^{(m\sigma+P/2,m;\ell)}_{[P];h_{[P],m}}$ with $|P|>2$ ($\sigma=\mathrm{sign}P$) that minimize[18] the conformal weight for a given spin m, namely

$$\Sigma^{(m\sigma+P/2,m;\ell)}_{[P];h_{[P],m}} = \sigma^{(m;\ell)}_{\sigma}\exp\left\{i\left(\sigma(m+\ell)+\tfrac{P}{2}\right)\varphi^1+\left(\sigma(m-\ell)+\tfrac{P}{2}\right)\varphi^2\right)\right\}\,, \tag{5.60}$$

$$h_{[P],m} = m(1-|P|)-P^2/4\,, \tag{5.61}$$

where we have suppressed the cocycle factor. These fields obey[19]

$$E(z)\Sigma^{(m\sigma+P/2,m;\ell)}_{[P];h_{[P],m}}(w) \;\sim\; \frac{m\sigma+P/2}{z-w}\Sigma^{(m\sigma+P/2,m;\ell)}_{[P];h_{[P],m}}(w)\,, \tag{5.62}$$

$$M_3(z)\Sigma^{(m\sigma+P/2,m;\ell)}_{[P];h_{[P],m}}(w) \;\sim\; \frac{\pm\ell}{z-w}\Sigma^{(m\sigma+P/2,m;\ell)}_{[P];h_{[P],m}}(w)\,, \tag{5.63}$$

$$J_{ij,0}\Sigma^{(m+P/2,m;\ell)}_{[P];h_{[P],m}}(z) \;=\; 0\,,\qquad P>0\,, \tag{5.64}$$

$$\bar{J}^{ij}_0\Sigma^{(-m+P/2,m;\ell)}_{[P];h_{[P],m}}(z) \;=\; 0\,,\qquad P<0\,, \tag{5.65}$$

corresponding to *massive ground states* in $\mathfrak{D}_{[P];h_{[P],m}}(m+P/2,m)\subset\mathfrak{D}_{[P]}$. We note that the special twist fields $\Sigma^{(P/2,0)}_{[P];h_{[P],0}}$, which realize P-twisted primary fields in the sense of (3.21)

[18] There also exist other massive fields in the model with higher conformal weights which will not be considered here. The maximal conformal weight for a given spin m is given by $h'_{[P],m}=m(|P|-1)-P^2/4$.

[19] The detailed fusion rules between generic massive representations are complicated, although bound to be of the form (3.59), given the validity of composition and distribution rules in (3.41) and (3.42).

and (3.22), obey conditions that are stronger than those in (5.64). We next expand the twist fields in modes,

$$\Sigma^{(e,m;\ell)}_{[P];h_{[P],m}}(z) = \sum_{l\in\mathbb{Z}-h_{[P],m}} \left(\Sigma^{(e,m;\ell)}_{[P];h_{[P],m}}\right)_l \; z^{-l-h_{[P],m}} \;, \tag{5.66}$$

where $e = m\sigma + P/2$. The corresponding states,

$$|[P]; h_{[P],m}; e, m; \ell\rangle \;=\; \lim_{z\to 0} \Sigma^{(e,m;\ell)}_{[P];h_{[P],m}}(z)|0\rangle \;=\; \left(\Sigma^{(e,m;\ell)}_{[P];h_{[P],m}}\right)_{-h_{[P],m}} |0\rangle \;, \tag{5.67}$$

thus describe singletons for $|P| = 1$, and massless as well as massive ground states with minimal conformal weight for $|P| > 1$. Inserting the mode expansions of the free fields yields (dropping a constant multiplicative factor)

$$|[P]; h_{[P],m}; e, m; \ell\rangle \;=\; \chi^{(m;\ell)}_\sigma e^{i[\sigma(m+\ell)+P/2]q^1 + i[\sigma(m-\ell)+P/2]q^2}|0\rangle \,, \tag{5.68}$$

where

$$\chi^{(m;\ell)}_+ \;=\; \xi^1_{-1}\cdots\xi^1_{-m-\ell}\xi^2_{-1}\cdots\xi^2_{\ell-m} \,, \quad \chi^{(m;\ell)}_- \;=\; \eta^1_{-1}\cdots\eta^1_{1-m-\ell}\eta^2_{-1}\cdots\eta^2_{1+\ell-m} \,. \tag{5.69}$$

As expected, the α^i_n oscillators drop out in the sector with minimal conformal weight. We also note that the free-field momenta $(p^1, p^2) = -(\sigma(m+\ell) + P/2, \sigma(m-\ell) + P/2)$ are related to energy eigenvalues and spin projections by the simple formulas $e = -(p^1 + p^2)/2$ and $\ell = -(p^1 - p^2)/2$. The twisted primary ground states $|[P]\rangle$ in (3.30) assume the particularly simple form[20]

$$|[P]\rangle \;=\; e^{\frac{iP}{2}(q^1+q^2)}|0\rangle \,, \tag{5.70}$$

with momentum $(p^1, p^2) = (-P/2, -P/2)$ and conformal weight $h = -p^i p^i/2 = -P^2/4$. These states obey the conditions in (4.40)–(4.43) by construction, with the affine generators realized in terms of the α^i_n and (ξ^i_n, η^i_n) oscillators. Thus, the spectral flow operation Ω_P acts on states (kets) by multiplication with $\exp(iP(q^1 + q^2)/2)$. This operation changes the energy eigenvalues and the conformal weights but clearly does not change the spin.

6. Conclusions and outlook

We have examined the $\widehat{\mathfrak{so}}(2, D-1)$ WZW model at the subcritical level $k = -(D-3)/2$. It has a singular vector, given by (3.12), at Virasoro level 2 in the NS sector whose decoupling induces a spectrum of KM twisted primary scalars $\Sigma_{[P]}$, $P \in \mathbb{Z}$, connected to the identity by P units of spectral flow and forming an operator algebra with fusion rule $\Sigma_{[P]} \times \Sigma_{[P']} = \Sigma_{[P+P']}$. The decoupling, or the hyperlight-likeness condition (3.10), constitutes an affine extension of the equation of motion of the $(D+1)$-dimensional conformal particle, i.e. the

[20] The dual states $\langle 0| = (|0\rangle)^\dagger$ obey $\langle 0|0\rangle = 1$, which induces the inner product $\langle [P']|[P]\rangle = \delta_{P+P',0}$, consistent with the definition in (3.57). The states given in (5.68) have dual representations given by $\langle [P]; h_{[P],m}; e, m; \ell| = (|[P]; h_{[P],m}; e, m; \ell\rangle)^\dagger$.

scalar singleton. For $P \neq 0$, the KM module built on $\Sigma_{[P]}$ contains a unitary subspace of representations in the tensor product of $|P|$ singletons or anti-singletons for $P > 0$ and $P < 0$, respectively. In the special case $D = 4$, we have shown that the spinor singleton and its composites also solve the decoupling condition (3.10). Moreover, in this case, by exploiting the isomorphism $\mathfrak{so}(2,3) \sim \mathfrak{sp}(4)$, we have considered the $\widehat{\mathfrak{sp}}(4)_{-1/2}$ WZW model admitting a realization in terms of 4 real-symplectic bosons, containing both scalar and spinor singletons together with their composites. A bosonization procedure leading to a free-field model has allowed us in particular to compute the fusion rules explicitly for $|P| \leq 2$ and compare with the predictions obtained by spectral flow arguments. We indeed find an agreement. These results provide an embedding of the Flato-Frønsdal compositeness theorem, stating that massless fields are made up from singletons in AdS, in a conformal field theory setting.

The massless sector $P = 2$ is particularly interesting for the purpose of making contact with higher-spin gauge theory. As noted below (5.34), the product between a singleton twist field and an anti-singleton twist field generates an element in the space $\mathcal{A} = Env(\mathfrak{sp}(4))/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the singular vacuum vector V_{AB} defined in (2.16). The space $\mathcal{A}$, which is an associative algebra, plays an important role in higher-spin gauge theory. In fact, higher-spin master fields are differential forms taking values in various subspaces of $\mathcal{A}$. In particular, the elements in $\mathcal{A}$ that are odd under the anti-automorphism τ define the higher-spin algebra $\mathfrak{hs}(4) \simeq \mathfrak{ho}_0(4,2)$. It would be interesting to spell out the exact relation between the higher-spin algebra and the affine algebra.

Another interesting observation deserving further investigation is related to the modular invariance of the theory and to the locality of the operator algebra. In fact, since $\Sigma_{[P]}$ are scalar particles, the corresponding vertex operators should have a local operator product with Grassmann even statistics, i.e. the monodromy matrix (5.13) should be equal to 1 for $\lambda = (P/2, P/2)$ and $\lambda' = (P'/2, P'/2)$. This would imply that $P \in 2\mathbb{Z}$. More generally, excited states in the P-twisted sector, created by N symplectic bosons, carry tensorial representations of Sp(4) for N even and spinorial representations for N odd. The locality properties of the operator product correspond to appropriate Grassmann statistics for $P, N \in 2\mathbb{Z}$. Interestingly, for these values also the conformal weight becomes an integer, which assures invariance under the modular transformation T without the need to include an anti-holomorphic sector. It would be interesting to perform a complete analysis of the locality of the vertex operator algebra.

We would also like to comment on similar realizations in terms of symplectic spinor bosons in $D = 5$ and $D = 7$. We expect these models to provide affine (massive) extensions of the massless models constructed in [35, 37]. Here, degeneracies among the states are lifted by internal gauge symmetries, based on U(1) and SU(2), respectively, in $D = 5$ and $D = 7$. In fact, as shown in [15], the affine extension is critical in $D = 7$ (where it is related by triality to a critical Sp(2) gauged model based on symplectic vector bosons). Studying these explicit realizations would be particularly interesting since, in principle, for special values of D extra singular vacuum vectors could appear, pointing to striking differences among models with different values of D. An example of this behavior in $D = 7$ will be discussed in a forthcoming paper [20].

JHEP02(2007)097

In [20], we will be mainly concerned with the gaugings of the $\widehat{\mathfrak{so}}(2, D-1)_{-\epsilon_0}$ WZW model. In fact, we expect that the gauging of a proper subalgebra $\mathfrak{h} \subset \mathfrak{g}$, such as that in (1.3), would remove all (or almost all) of the non-zero mode excitations in the spectrum. We would then be left with a topological model containing in its spectrum scalar (and, in $D = 4$, spinor) (anti)singletons as well as all of their composites. The gauging, however, would not necessarily remove all non-unitary states. In particular, in $D = 4$ the non-unitary anti-spinor singleton zero-mode representation $\mathfrak{D}^-_{1/2}$ would survive. The truncation of this representation, along with other representations with the wrong space-time statistics, should be controlled by a GSO-like projection. We expect the resulting GSO-projected gauged WZW model to be unitary and to consist of scalar singleton and anti-singleton representations together with all of their tensor products.

Finally, we would like to say a few words on how a proper gauging of the WZW models discussed in this paper could fit in a bigger picture hopefully providing an alternative approach to string quantization in AdS spacetime. As mentioned in the Introduction, an alternative to the conventional interpretation of the target space as a space-time manifold is to regard it as an internal fiber [15]. The WZW model would then be thought of as a device to realize the internal symmetry algebra, together with a star product and a trace operation. The spacetime would appear only upon the operation of unfolding [13]. A crucial test of this idea would then be to set the issue whether unfolding could be implemented consistently in conjunction with the WZW model and, if so, whether the correct free space-time field equations [51] for the fields in the model could be reproduced. Our work is a at very preliminary stage and the understanding of these issues is left for future work.

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A. Decomposition formulae for $P = 0, 1$

In this appendix we shall derive the decomposition (4.46) of the P-twisted oscillator Fock spaces $\widehat{\mathcal{F}}_{[P]}$ into irreps $\widehat{\mathfrak{D}}_{[P]}(e_0, s_0)$ of $\widehat{\mathfrak{so}}(2,3)_{-1/2}$. The decompositions for different values of P are related by spectral flow, which means that it suffices to show (4.46) for one value of P. In this appendix we shall do this for $P = 1$ and $P = 0$ using oscillator methods (without resorting to spectral flow), viewing one of the cases as a check of the spectral flow

formalism. In general, the Fock spaces factorize

$$\widehat{\mathcal{F}}_{[P]} = \widehat{\mathcal{F}}^{(1)}_{[P]} \otimes \widehat{\mathcal{F}}^{(2)}_{[P]} \ , \tag{A.1}$$

where $\widehat{\mathcal{F}}^{(i)}_{[P]}$ generated by the action of $a^i(z)$ and $a_i(z)$ on $|[P]\rangle$ for fixed $i = 1, 2$. The $\widehat{\mathfrak{so}}(2,3)_{-1/2}$ irreps decompose in a corresponding way under $\widehat{\mathfrak{so}}(2,3)_{-1/2} \to \widehat{\mathfrak{sp}}(2)^{(1)}_{-1/2} \oplus \widehat{\mathfrak{sp}}(2)^{(2)}_{-1/2}$, where the affine $sp(2)$ currents are given in (4.14). Thus, in order to decompose $\widehat{\mathcal{F}}_{[P]}$ under $\widehat{\mathfrak{so}}(2,3)_{-1/2}$ we can first decompose $\widehat{\mathcal{F}}^{(i)}_{[P]}$ under $\widehat{\mathfrak{sp}}(2)^{(i)}_{-1/2}$ and then examine the action of the off-diagonal $\widehat{\mathfrak{so}}(2,3)_{-1/2}$ currents on the tensor product.

Case $P = 1$

Lemma 1. For fixed i, the Fock space $\widehat{\mathcal{F}}^{(i)}_{[1]}$ decomposes under $\widehat{\mathfrak{sp}}(2)^{(i)}_{-1/2}$ as

$$\widehat{\mathcal{F}}^{(i)}_{[1]} = \widehat{\mathfrak{D}}^{(i)}_{[1]}(\tfrac{1}{4}) \oplus \widehat{\mathfrak{D}}^{(i)}_{1]}(\tfrac{3}{4}) \ , \tag{A.2}$$

where $\widehat{\mathfrak{D}}^{(i)}_{[1]}(j)$ are built on the two metaplectic representations $\mathfrak{D}(j)$ of $\mathfrak{sp}(2)$ with $j = 1/4, 3/4$,

$$J^+_n|j\rangle^{(i)} = 0 \quad \text{for } n \geq 1 \, , \tag{A.3}$$

$$(J^3_n - j\delta_{n0})|j\rangle^{(i)} = 0 \quad \text{for } n \geq 0 \ , \tag{A.4}$$

$$J^-_n|j\rangle^{(i)} = 0 \quad \text{for } n \geq -1 \ . \tag{A.5}$$

We note that $\mathfrak{D}(1/4)$ and $\mathfrak{D}(3/4)$ are isomorphic to the even and odd states, respectively, of the Fock space of a single oscillator (in (A.5), the J^-_{-1} condition removes a singular vector that is identically zero in the oscillator realization).

Lemma 2. The off-diagonal $\widehat{\mathfrak{sp}}(4)_{-1/2}$ charges $\{\bar{J}^{12}_m, K^{12}_n, K^{21}_n, J^{12}_n\}$ act on tensor products as follows:

$$\widehat{\mathfrak{D}}^{(1)}_{[1]}(j) \otimes \widehat{\mathfrak{D}}^{(2)}_{[1]}(j') \mapsto \widehat{\mathfrak{D}}^{(1)}_{[1]}(1-j) \otimes \widehat{\mathfrak{D}}^{(2)}_{[1]}(1-j') \ . \tag{A.6}$$

It follows that $\widehat{\mathfrak{sp}}(4)_{-1/2}$ acts irreducibly on

$$\widehat{\mathfrak{D}}_{[1]}(\tfrac{1}{2}, 0) = \left[\widehat{\mathfrak{D}}^{(1)}_{[1]}(\tfrac{1}{4}) \otimes \widehat{\mathfrak{D}}^{(2)}_{[1]}(\tfrac{1}{4})\right] \oplus \left[\widehat{\mathfrak{D}}^{(1)}_{[1]}(\tfrac{3}{4}) \otimes \widehat{\mathfrak{D}}^{(2)}_{[1]}(\tfrac{3}{4})\right] \ , \tag{A.7}$$

$$\widehat{\mathfrak{D}}_{[1]}(1, \tfrac{1}{2}) = \left[\widehat{\mathfrak{D}}^{(1)}_{[1]}(\tfrac{1}{4}) \otimes \widehat{\mathfrak{D}}^{(2)}_{[1]}(\tfrac{3}{4})\right] \oplus \left[\widehat{\mathfrak{D}}^{(1)}_{[1]}(\tfrac{3}{4}) \otimes \widehat{\mathfrak{D}}^{(2)}_{[1]}(\tfrac{1}{4})\right] \ . \tag{A.8}$$

We note that the ground state of $\widehat{\mathfrak{D}}_{[1]}(\frac{1}{2}, 0)$ is given by $|1/4\rangle^{(1)} \otimes |1/4\rangle^{(2)}$, while $|3/4\rangle^{(1)} \otimes |3/4\rangle^{(2)} = \bar{J}^{12}_0|1/4\rangle^{(1)} \otimes |1/4\rangle^{(2)}$.

Combining Lemmas 1 and 2 with the factoring formula (A.1) we conclude that $\widehat{\mathcal{F}}_{[1]}$ decomposes into the direct sum of (A.7) and (A.8), as stated in (4.46) for $P = 1$.

Derivation of Lemma 1: To simplify the notation, let us drop the (i) superscripts and define $(\bar{a}_n, a_n) = (\bar{a}^i_n, a_{i,n})$ and $\widehat{\mathcal{F}} = \widehat{\mathcal{F}}^{(i)}_{[1]}$. We decompose $\widehat{\mathcal{F}}$ into Virasoro levels,

$$\widehat{\mathcal{F}} = \bigoplus_{\ell=0}^{\infty} \widehat{\mathcal{F}}_\ell \,, \qquad (L_0 - \ell + \tfrac{1}{8})\widehat{\mathcal{F}}_\ell \;=\; 0 \;. \tag{A.9}$$

We seek the decomposition into irreducible $\widehat{\mathfrak{sp}}(2)_{-1/2}$ representations,

$$\widehat{\mathcal{F}} = \bigoplus_{\ell,j} \widehat{\mathfrak{D}}_\ell(j) \,, \tag{A.10}$$

where $\widehat{\mathfrak{D}}_\ell(j)$ is built on the ground state $|\ell; j\rangle$ at level ℓ with $\mathfrak{sp}(2)$ spin j. There are two ground states at level 0, namely

$$|0; \tfrac{1}{4}\rangle \;=\; |[1]\rangle \,, \qquad |0; \tfrac{3}{4}\rangle \;=\; \bar{a}_0|[1]\rangle \;. \tag{A.11}$$

Let us define

$$\widehat{\mathcal{M}} = \widehat{\mathfrak{D}}_0(\tfrac{1}{4}) \oplus \widehat{\mathfrak{D}}_0(\tfrac{3}{4}) \;=\; \bigoplus_{\ell} \widehat{\mathcal{M}}_\ell \,, \tag{A.12}$$

and consider the spaces $\widehat{\mathcal{Q}}_\ell = \widehat{\mathcal{F}}_\ell / \widehat{\mathcal{M}}_\ell$. By definition, $\widehat{\mathfrak{D}}_\ell(j)$ does not contain any singular vectors, so that if $|\ell\rangle$ is a ground state at level $\ell \geq 1$, then $|\ell\rangle \in \widehat{\mathcal{Q}}_\ell$. Thus, if $\dim \widehat{\mathcal{Q}}_\ell = 0$ for $\ell \geq 1$ then there are no ground states for $\ell \geq 1$. Conversely, if there are no ground states for $\ell \geq 1$ then $\dim \widehat{\mathcal{Q}}_\ell = 0$ for $\ell \geq 1$ (since the first occurrence of $\dim \widehat{\mathcal{Q}}_\ell > 0$ would be tied to the existence of such a ground state). Hence, $\dim \widehat{\mathcal{Q}}_\ell = 0$ for $\ell \geq 1$ is equivalent to that there are no ground states for $\ell \geq 1$. This can be checked explicitly for $\ell \leq 3$ (either by showing there are no ground states or by simply rearranging oscillator excitations into the form of KM descendants in $\widehat{\mathcal{M}}$). We proceed by induction. A state $|\ell; N\rangle \in \widehat{\mathcal{F}}_\ell$ with fixed affine $\mathfrak{sp}(2)$ spin, say $(J^3_0 - \frac{N}{2} - \frac{1}{4})|\ell; N\rangle = 0$ ($N \in \mathbb{Z}$), can be expanded as

$$|\ell; N\rangle \;=\; \sum_{\substack{\{m\},\{n\} \\ \sum m + \sum n = \ell \\ N' \equiv N - \sum_m 1 + \sum_n 1 \geq 0}} A_{\{m\},\{n\}} \prod_m \bar{a}_{-m} \prod_n a_{-n} |N'\rangle \,, \tag{A.13}$$

where $A_{\{m\},\{n\}}$ are constants, $\{m\}$ and $\{n\}$ are sets of positive integers, and $|N'\rangle = (a_0)^{N'}|[1]\rangle$ belong to $\mathfrak{D}(1/4)$ and $\mathfrak{D}(3/4)$ for N' even and odd, respectively. In each monomial, at least one m or n must be positive. Suppose $m > 0$. By the induction assumption, the monomial can then be rewritten as $\bar{a}_{-m} \sum_{\{k,\alpha\}} \prod_{k,\alpha} J^\alpha_{-k} |N''\rangle$ where k are positive integers and N'' is fixed by spin conservation. By moving the KM charges to the left, the monomial can be rearranged into $a_{-\ell}$ and $\bar{a}_{-\ell}$ excitations plus descendants in $\widehat{\mathcal{M}}_\ell$. An analogous statement holds if $n > 0$. Thus,

$$|\ell; N\rangle \;=\; A\bar{a}_{-\ell}|N-1\rangle + Ba_{-\ell}|N+1\rangle + |\ell; N; \mathrm{desc}\rangle \,, \qquad |\ell; N; \mathrm{desc}\rangle \in \widehat{\mathcal{M}}_\ell \,, \tag{A.14}$$

where A and B are constants (we note that this shows that $\dim \widehat{\mathcal{Q}}_\ell \leq 2$), so that $\dim \widehat{\mathcal{Q}}_\ell = 0$ if

$$J_1^\alpha |\ell; N\rangle = 0 \;\Rightarrow\; A = B = 0\,. \tag{A.15}$$

We note that the absence of singular vectors in M_ℓ assures that for fixed A and B the constraint $J_1^\alpha|\ell;N\rangle = 0$ has a unique solution for $|N,\ell;\mathrm{desc}\rangle$ (which may of course be trivial), since if $|N,\ell;\mathrm{desc}\rangle'$ is another solution then $J_1^\alpha(|N,\ell;\mathrm{desc}\rangle - |N,\ell;\mathrm{desc}\rangle') = 0$ implies $|N,\ell;\mathrm{desc}\rangle - |N,\ell;\mathrm{desc}\rangle' = 0$. We now expand

$$|N,\ell;\mathrm{desc}\rangle = \sum_{k=2}^{\ell} |N,\ell;(k)\rangle\,, \tag{A.16}$$

where $|N,\ell;(k)\rangle$ is k'th order in oscillators $(a_{-n},\bar{a}_{-n})$ with $n \geq 1$. The strategy is to work order by order in k and arrive at some finite order at $A = B = 0$ as a compatibility condition. Canceling the linear terms in the J_1^+ and J_1^- conditions yields

$$|N,\ell;(2)\rangle = -B\bar{a}_{-\ell+1}a_{-1}|N\rangle - \frac{A}{N}a_{-\ell+1}\bar{a}_{-1}|N\rangle\,. \tag{A.17}$$

Canceling the linear terms in the J_1^3 condition then yields the compatibility condition

$$A + NB = 0\,. \tag{A.18}$$

In the next order one finds

$$|N,\ell;(3)\rangle = \frac{A}{N}\bar{a}_{-\ell+2}\bar{a}_{-1}a_{-1}|N-1\rangle + \frac{B}{N+1}a_{-\ell+2}a_{-1}\bar{a}_{-1}|N+1\rangle\,, \tag{A.19}$$

and the compatibility conditions (for $\ell > 3$)

$$A - NB = 0\,, \qquad \frac{N-1}{N}A = 0\,, \qquad \frac{1}{N+1}B = 0\,, \tag{A.20}$$

implying $A = B = 0$, which completes the proof of Lemma 1.

Case $P = 0$. The analysis parallels that of $P = 1$. Let us assume the decomposition

$$\widehat{\mathcal{F}}^{(i)}_{[0]} = \widehat{\mathfrak{D}}^{(i)}_{[0]}(0) \oplus \widehat{\mathfrak{D}}^{(i)}_{[0]}(-\tfrac{1}{2})\,, \tag{A.21}$$

with $\widehat{\mathfrak{sp}}(2)^{(i)}_{-1/2}$ ground states given by the singlet $|[0]\rangle$ and the doublet $(a^i_{-1/2}|[0]\rangle, \bar{a}^i_{-1/2}|[0]\rangle)$ (fixed i), with lowest $\mathfrak{sp}(2)^{(i)}$ spins $j = 0$ and $j = -1/2$, respectively. The off-diagonal $\widehat{\mathfrak{sp}}(4)_{-1/2}$ charges then act on tensor products as follows

$$\mathfrak{D}^{(1)}_{[0]}(j) \otimes \mathfrak{D}^{(2)}_{[0]}(j') \mapsto \mathfrak{D}^{(1)}_{[0]}(-\tfrac{1}{2}-j) \otimes \mathfrak{D}^{(2)}_{[0]}(-\tfrac{1}{2}-j')\,, \tag{A.22}$$

implying that $\widehat{\mathfrak{sp}}(4)_{-1/2}$ acts irreducibly on

$$\widehat{\mathfrak{D}}_{[0]}(0,0) = \left[\mathfrak{D}^{(1)}_{[0]}(0) \otimes \mathfrak{D}^{(2)}_{[0]}(0)\right] \oplus \left[\mathfrak{D}^{(1)}_{[0]}(-\tfrac{1}{2}) \otimes \mathfrak{D}^{(2)}_{[0]}(-\tfrac{1}{2})\right]\,, \tag{A.23}$$

$$\widehat{\mathfrak{D}}_{[0]}(1,\tfrac{1}{2}) = \left[\mathfrak{D}^{(1)}_{[0]}(0) \otimes \mathfrak{D}^{(2)}_{[0]}(-\tfrac{1}{2})\right] \oplus \left[\mathfrak{D}^{(1)}_{[0]}(-\tfrac{1}{2}) \otimes \mathfrak{D}^{(2)}_{[0]}(0)\right]\,. \tag{A.24}$$

Combined with the factoring formula (A.1), this yields (4.46) for $P = 0$. Finally, to derive (A.21) we decompose into Virasoro levels $\ell \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$. The first two levels contain the ground states, and there are no new ground states at levels $\ell = 1$ and $\ell = 3/2$. We proceed by induction on ℓ, treating $\ell \in \mathbb{Z} + 1/2$ and $\ell \in \mathbb{Z}$ as separate cases. The steps are similar to those for $P = 1$. For example, in case $\ell \in \mathbb{Z} + 1/2$, the induction assumption implies $|\ell\rangle = (Aa_{-\ell} + B\bar{a}_{-\ell})|0\rangle + |\ell; \text{desc}\rangle$ where $|\ell; \text{desc}\rangle = \sum_{k\geq 1} |\ell; (2k+1)\rangle \in \widehat{\mathfrak{D}}_{[0]}(-\frac{1}{2})$. The J_1^+ conditions then yield

$$|\ell; (3)\rangle = (-Aa_{-\ell+1}(\bar{a}_{-1/2})^2 + B\bar{a}_{-\ell+1}(a_{-1/2})^2)|0\rangle\,, \tag{A.25}$$

which is annihilated by J_1^3 to lowest order in oscillators only for $A = B = 0$ (for $\ell \geq 3/2$).

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JHEP02(2007)097

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JHEP02(2007)097

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JHEP02(2007)097